EML 4312: Control of Mechanical Engineering Systems

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Solving Linear Dynamic Equations

In the previous set of notes, we reduced nonlinear dynamic systems to linear forms. In particular, we were interested in obtaining the linear approximation around the steady-state reference, $\mathbf{x}^*_{ss}(t) = \mathbf{0}$. The resulting equation turns out to be

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{1}$$

Note that we have now added in the control term, also in linear form. In these notes, we will try and solve the above equation.

Time Domain Solution

Consider the scalar case ($n = m = 1$)

$$\dot{x} = ax + bu \tag{2}$$

with the initial condition, $x(t = 0) = x_0$. The above equation integrates out to give

$$x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)}\, b\, u(\tau)\, d\tau \tag{3}$$

where, τ is a dummy integration variable. For the vector case ($n \geq 1$, $m \geq 1$, given in Eq.(1)), the integration looks like

$$\mathbf{x}(t) = e^{\mathbf{A}\,t}\mathbf{x}_0 + \int_0^t e^{A(t-\tau)}\, \mathbf{B}\, \mathbf{u}(\tau)\, d\tau \tag{4}$$

In the above equation, the term $e^{\mathbf{A}t}$ is very interesting and is called a *matrix exponential*. It is a matrix of size $n \times n$ and has a beautiful analogy with the exponential of a scalar:

$$e^{at} = 1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \ldots \tag{5a}$$

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \mathbf{A}^3\frac{t^3}{3!} + \ldots \tag{5b}$$

where $\mathbf{I}$ is a $n \times n$ identity matrix. The analogy ends there because unlike the scalar case, there is no easy way of evaluating the matrix exponential. MATLAB©has an inbuilt function called "expm" for computing this term. The classic paper by Moler and Van Loan is highly recommended on this topic.[1]

[1] C. Moler and C. Van Loan, "Nineteen Dubious Ways to Compute the Exponential of a Matrix: Twenty-Five Years Later", *SIAM Review*, Vol. 45, No.1, pp 3-49, 2003 (originally published in the *SIAM Review* in 1978. **Abstract of the 2003 reprise:** *In principle, the exponential of a matrix could be computed in many ways. Methods involving approximation theory, differential equations, the matrix eigenvalues, and the matrix characteristic polynomial have been proposed. In practice, consideration of computational stability and efficiency indicates that some of the methods are preferable to others but that none are completely satisfactory. Most of this paper was originally published in 1978. An update, with a separate bibliography, describes a few recent developments.*

Note that the matrix exponential term is also very interesting because it represents a "map" that maps the initial conditions ($\mathbf{x}_0$) into the current conditions ($\mathbf{x}(t)$).. and for this reason, it is also called the *state transition matrix*. This is especially clear in the absence of control, where the solution simply becomes

$$\underbrace{\mathbf{x}(t)}_{\text{current}} = \underbrace{e^{\mathbf{A}\,t}}_{\Phi(t,t_0)} \underbrace{\mathbf{x}_0}_{\text{initial}} \tag{6}$$

So we can write Eq.(4) alternatively as

$$\mathbf{x}(t) = \Phi(t,t_0)\mathbf{x}_0 + \int_0^t e^{A(t-\tau)}\,\mathbf{B}\,\mathbf{u}(\tau)\,d\tau \tag{7}$$

We will not spend any more time on the time-domain solution of linear equations.

Laplace Transforms

We all know it is much easier to solve algebraic equations than ordinary differential equations. The mathematical construct called the *Laplace transform* allows us to do just that.. it converts equations of the type in Eq.(1) into a system of algebraic equations, solving which involves nothing more than finding roots of polynomials. This is much more desirable that evaluating matrix exponentials and integrals! So, we will study how Laplace transforms can be used to solve linear ODEs.

- As always, we begin with the scalar case. Consider the scalar signal (or function of time) $f(t)$. Its Laplace transform is defined as

$$\mathcal{L}\{f(t)\} \equiv F(s) \triangleq \int_{0^-}^{\infty} f(t)\,e^{-st}\,dt \tag{8}$$

  where, s is a complex number, $s = \sigma + j\omega$, where again, $j = \sqrt{-1}$.

  Essentially, we are making a change of variables, exchanging t for s! We are going from the *time-domain* to the s-domain, which is also called the *frequency-domain*:

$$\underset{\text{Time Domain}}{t} \xrightarrow{\mathcal{L}\{\cdot\}} \underset{\text{Frequency Domain}}{s} \tag{9}$$

  The enabler for this change of variables is the Laplace transform. In order to return to the time-domain from the frequency domain, we can employ the **inverse** Laplace transform, defined as:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \triangleq \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)\,e^{st}\,ds \tag{10}$$

  This forward and backward transformation is shown in Fig.(1)

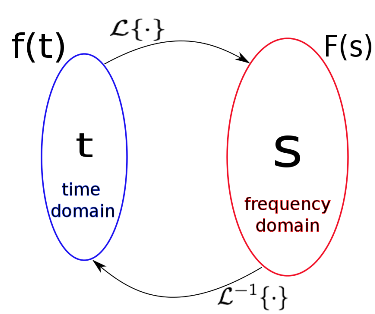


Figure 1: The Laplace Transformation Between Time & Frequency Domains

Notes:

1. The lower limit in the integral of Eq.(8) can be understood as

$$0^- \triangleq \lim_{\delta \downarrow 0} (0 - \delta) \tag{11}$$

   In other words, the integration begins "just to the left" of the time $t = 0$, thereby accounting for a potential jump discontinuity in the function $f(t)$ at $t = 0$.

2. At the outset, it appears like we are complicating matters! We have exchanged a real-valued function in the time domain ($f(t)$) for a complex valued function ($F(s)$). This was done by via the forward Laplace transform in Eq.(8), which essentially amounts to "integrating-out", or, "averaging-out" time. Similarly, the inverse Laplace transform is equivalent to integrating out, or, averaging out the signal $F(s)$ over the entire range of frequencies, from $\omega = -\infty$ to $\omega = +\infty$.

   As we continue with these developments, you will begin to realize that Laplace transforms are indeed useful. They will enable us to convert ODEs in the time-domain to algebraic equations in the frequency-domain. The latter will be much easier to solve. Then, the inverse transform will allow us to reconstruct the solution in our physically appealing time-domain.

3. The Laplace transform involves an integration over a very long time period! As a result, it may not always exists (e.g. it may blow up to infinity). The *condition for existence* of the Laplace transform of $f(t)$ is:

   The Laplace transform of the time-signal $f(t)$ exists if $\exists$ (there exists) a $\sigma_1 > 0 \in \Re$ such that

$$\int_0^t |f(t)| \, e^{-\sigma_1 t} \, dt < \infty \tag{12}$$

   where, $\Re$ is the set of real numbers.

4. **Region of convergence.** It is possible that the Laplace transform is not defined for all values of the complex variable s. The entire set of values of s for which the Laplace transform $F(s)$ of $f(t)$ is defined is called the *region of convergence* (ROC) of $F(s)$.

5. In the inverse Laplace transform (Eq.(10)), the real number σ is allowed to be anything so long as $(\sigma + j0)$ lies inside the ROC of $F(s)$. Note that this ties in nicely with item #3 above.

Definition 1. ***Poles of*** $F(s)$. *The values of "s"' for which either $F(s)$ or any of its derivatives (with respect to s) approach ∞.*

❣ **Example** Let $F(s) = \frac{1}{s+1}$. $F(s)$ has only one pole, namely, $p = -1 + j0$.

Definition 2. *A pole "p" is a pole of **order** n if $F(s)\,(s-p)^n$ is a finite, nonzero number.*

❣ **Example** Let $F(s) = \frac{s+1}{(s+2)^2}$. Clearly, note that $p = -2$ is a pole of $F(s)$ because $\lim_{s \to -2} F(s) = \infty$. Moreover, at $s = -2$,

$$\begin{aligned}
F(s)(s+2)^0 &= \frac{-1}{0.0} = \infty \\
F(s)(s+2)^1 &= \frac{-1}{0} = \infty \\
F(s)(s+2)^2 &= -1 \quad \text{(finite and nonzero!)} \\
F(s)(s+3)^3 &= -1.0 = 0 \\
F(s)(s+4)^4 &= -1.0.0 = 0 \\
&\cdots
\end{aligned}$$

So, $F(s)(s-p)^n$ is finite and nonzero only for $n = 2$, whereby the order of pole $p = -2$ is 2.

Definition 3. *A pole of order $n = 1$ is called a* ***simple pole****.*

- Laplace transform of some common functions:

  1. **Impulse function**

  $$\delta(t) = \begin{cases} \lim_{t_0 \downarrow 0} \frac{A}{t_0} & t \in [-t_0, t_0] \\ 0 & \text{o.w. (otherwise)} \end{cases} \tag{13}$$

  This function is essentially a spike at $t = 0$ and zero elsewhere. The "magnitude" of the spike is A. It is shown in Fig.(2(a)). The Laplace transform is:

  $$\mathcal{L}\{\delta(t)\} = A \tag{14}$$

  2. **Unit step function**.

  $$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \tag{15}$$

  This is depicted in Fig.(2(b)). This function is "off" until $t = 0$, after which it "turns on" and stays constant at unity. Its Laplace transform is

  $$\mathcal{L}\{u(t)\} = \frac{A}{s} \tag{16}$$

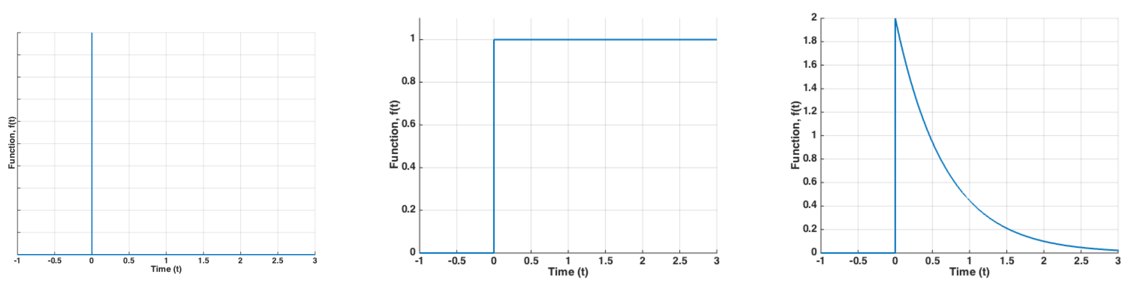
  ⊛ It is interesting to note that if we multiply any function $f(t)$ with $u(t)$, we will essentially chop everything off of $f(t)$ to the left of $t = 0$. Several examples are shown below.

  3. **Exponential function**

  $$f(t) = Au(t)e^{-at}, \quad a > 0 \tag{17}$$

  As mentioned above, the effect of multiplying the exponential function with the unit-step function is to hammer everything to the left of $t = 0$ down to zero. This is shown in Fig.(2(c)). The Laplace transform is:

  $$\mathcal{L}\{f(t)\} = \frac{A}{s + a} \tag{18}$$



(a) Impulse function $\delta(t)$ at $t = 0$ of magnitude A

(b) Unit step function, "stepping up" (turning on) at $t = 0$

(c) Exponential function in $t \geq 0$

Figure 2: Some common time-domain signals

  4. **Ramp function**

  $$f(t) = Au(t)t \tag{19}$$

  The ramp function is linear, shown in Fig.(3(a)). Its Laplace transform is:

  $$\mathcal{L}\{f(t)\} = \frac{A}{s^2} \tag{20}$$

5. **Monomials**

$$f(t) = Au(t)t^k, \quad k = 1, 2 \ldots \tag{21}$$

The monomial function is a generalization of the ramp function and is nonlinear for $k > 1$. For example, the quintic monomial ($k = 5$) is shown in Fig.(3(b)). Its Laplace transform is:

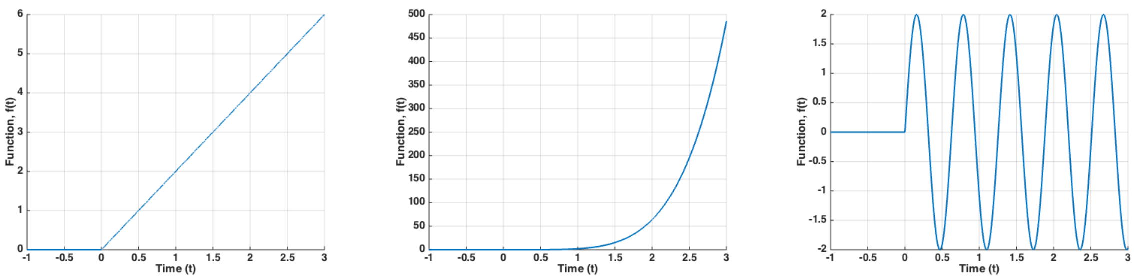
$$\mathcal{L}\{f(t)\} = A\frac{k!}{s^{k+1}} \tag{22}$$

6. **Sinusoids**

$$f(t) = Au(t)\sin\omega t \tag{23}$$

This is shown in Fig.(3(c)). Its Laplace transform is:

$$\mathcal{L}\{f(t)\} = A\frac{\omega}{s^2 + \omega^2} \tag{24}$$



(a) Ramp function, $f(t) = Au(t)t$ (b) Monomial, $f(t) = Au(t)t^k$ (c) Sinusoid, $f(t) = Au(t)\sin\omega t$

Figure 3: Some more common time-domain signals

- Useful properties of the Laplace transform.

1. **Linearity.** The Laplace transformation (LT) is a linear operation! (not a big surprise because all it involves is an integration, which is a linear operation). In other words, for time signals $f_1(t)$ and $f_2(t)$ and constant scalars a and b,

$$\mathcal{L}\{af_1(t) + bf_2(t)\} = a\mathcal{L}\{f_1(t)\} + b\mathcal{L}\{f_2(t)\} \tag{25}$$

2. **Function translations.** Given the LT of a time-signal, $f(t)$, we can derive the LT of some of its simple variations. Let $\mathcal{L}\{f(t\} = F(s)$. We have:

(a) **Time delay.** Let $g(t)$ be the time-delayed signal $f(t)$ by a units of time:

$$g(t) = f(t - a) \tag{26}$$

A signal and its delayed version (by 0.2 seconds) is shown in Fig.(4). The LT of $g(t)$ is

$$G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{f(t - a)\} = e^{-as}F(s) \tag{27}$$

(b) **Damping.** Let $g(t)$ be the damped signal $f(t)$, such that:

$$g(t) = e^{-at}f(t), \quad a > 0 \tag{28}$$

Then, the LT of $g(t)$ is

$$G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{e^{-at}f(t)\} = F(s + a) \tag{29}$$

An example is shown in Fig.(5) where, $f(t) = 2\sin\omega t$ and $g(t) = 2e^{-t}\sin\omega t$. Then, following Eqs.(24) and (29),

$$G(s) = \frac{2\omega}{(s + 1)^2 + \omega^2} \tag{30}$$

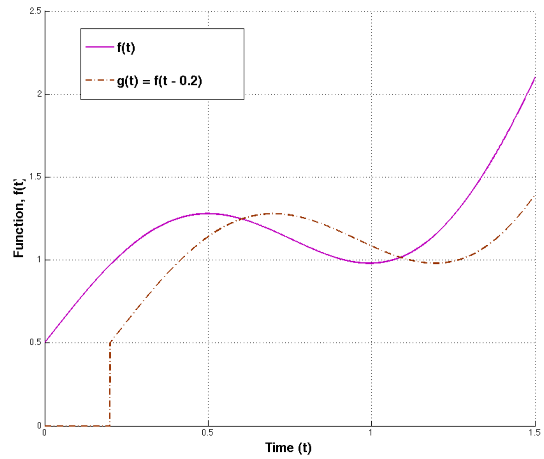


Figure 4: Time-Delayed Signal

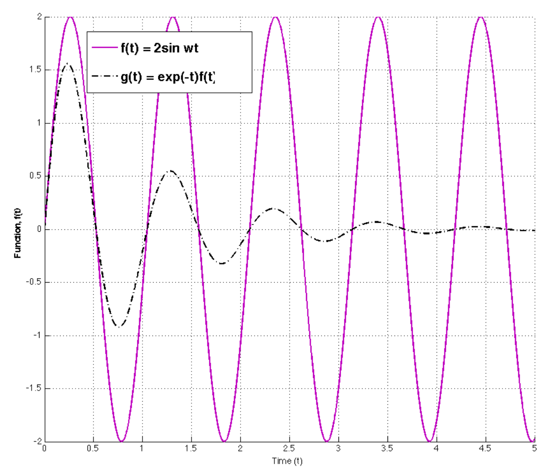


Figure 5: Damped Signal

(c) **Time scaling.** Let $g(t)$ be a time-scaled version of $f(t)$ (either sped-up or slowed-down):

$$g(t) = f(t/a), \quad a > 0 \tag{31}$$

Clearly, if $a > 1$, the function $g(t)$ is a slowed-down version of $f(t)$. On the other hand, $0 < a < 1$ represents a speeding-up by a factor $1/a$. Then the LT of $g(t)$ is

$$G(s) = a\,F(as) \tag{32}$$

An example is shown in Fig.(6), where, $f(t) = 2\sin\omega t$ and $g(t) = 2\sin 2\omega t$, such that $a = 0.5$. You can clearly see that the $g(t)$ signal has twice the frequency of the $f(t)$ signal, i.e. it is twice as fast as $f(t)$. We have

$$G(s) = \frac{1}{2}\frac{2\omega}{\frac{s^2}{4} + \omega^2} = \frac{4\omega}{s^2 + 4\omega^2} \tag{33}$$

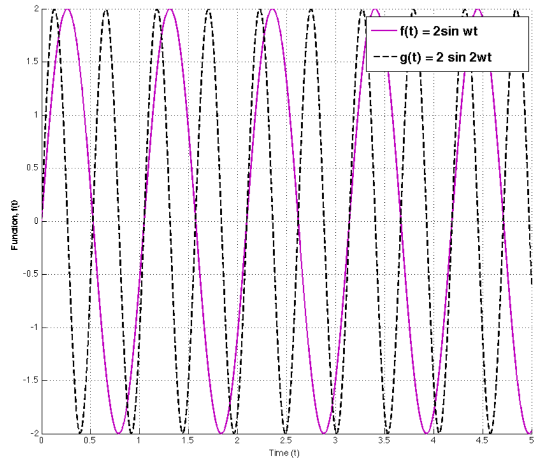
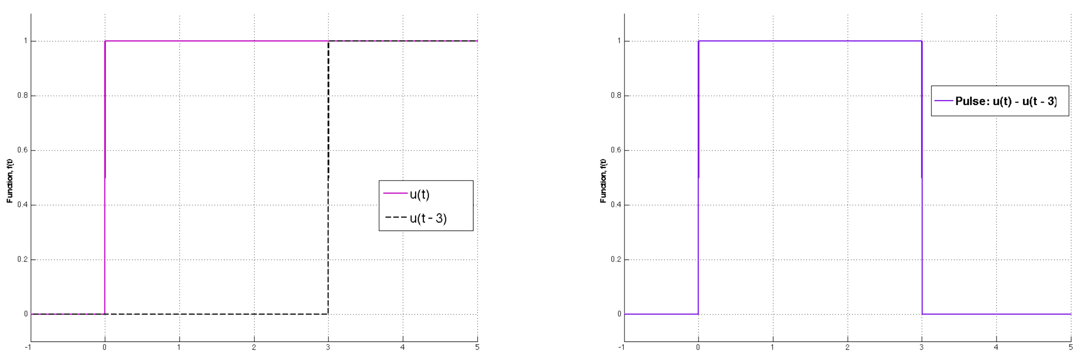


Figure 6: Time-Scaled Signal

❣ **Example Pulse function.** Consider the pulse function, which is defined as

$$f(t) = \begin{cases} A/a, & t \in [0, a] \\ 0, & \text{o.w.} \end{cases} \tag{34}$$

The pulse is said to have an amplitude of A/a. It is interesting to note that the pulse function can be constructed by taking the difference between a step function and its time-delayed variation. This is shown in Fig.(7), where, $f(t) = u(t) - u(t-3)$ creates a pulse of unit amplitude and width $= 3$ *seconds*.



(a) A step-function and its time-delayed version (delay = 3 s)

(b) The difference of the two, resulting in a pulse-function

Figure 7: The pulse function as the difference of a step function and its time-delayed variation

In general,

$$f(t) = \frac{A}{a}(u(t) - u(t-a)) \tag{35}$$

The above relationship is very useful because it is now easy to compute the Laplace transform of a pulse using properties 1 (linearity) and 2.(a) (time-delay):

$$F(s) = \frac{A}{a}\Big(\mathcal{L}\{u(t)\} - \mathcal{L}\{u(t-a)\}\Big) \tag{36a}$$

$$= \frac{A}{a}\left(\frac{1}{s} - \frac{e^{-as}}{s}\right) = \frac{A}{as}(1 - e^{-as}) \tag{36b}$$

- More useful properties of the Laplace transform (continued from above). Let $\mathcal{L}\{f(t)\} = F(s)$.

3. **Differentiation.** Differentiation can occur in the time-domain (wrt t) or in the frequency domain (wrt s). We consider both case:
   (a) **Real Differentiation.** Let $g(t) = \frac{df}{dt}$. Then,

$$\mathcal{L}\{g(t)\} = G(s) = \mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0) \tag{37}$$

   In other words, a differentiation in the time-domain amounts to a simple multiplication with s in the frequency domain.

   (b) **Corollary (a).1.**

$$\mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\} = s^2 F(s) - sf(0) - \dot{f}(0) \tag{38}$$

   This is easy to prove using the basic result in Eq.(37)..

$$\text{LHS} = \mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\} = \mathcal{L}\left\{\frac{d}{dt}\left(\frac{df}{dt}\right)\right\} \tag{39a}$$

$$\underset{\text{Eq.37}}{=} s\mathcal{L}\left\{\frac{df}{dt}\right\} - \frac{df}{dt}(0) \tag{39b}$$

$$\underset{\text{Eq.37}}{=} s\left(s\mathcal{L}\{f(t)\} - f(0)\right) - \dot{f}(0) \quad \blacksquare \tag{39c}$$

   (c) **Corollary (a).2.**

$$\mathcal{L}\left\{\frac{d^k f}{dt^k}\right\} = s^k F(s) - s^{k-1}f(0) - s^{k-2}\dot{f}(0) - \ldots - sf^{k-2}(0) - f^{k-1}(0) \tag{40}$$

   where, $f^p(0) \triangleq \frac{d^p f}{dt^p}(0)$.

   (d) **Complex differentiation.** (in the frequency domain). Let $F(s) = \mathcal{L}\{f(t)\}$. Then,

$$\frac{dF(s)}{ds} = \mathcal{L}\{tf(t)\} \tag{41}$$

   everywhere, except at the poles of $F(s)$.

   (e) **Corollary (d.1).**

$$\frac{d^2 F(s)}{ds^2} = \mathcal{L}\{t^2 f(t)\} \tag{42}$$

   (f) **Corollary (d.2).**

$$\frac{d^k F(s)}{ds^k} = \mathcal{L}\{t^k f(t)\} \tag{43}$$

   ❣ **Example** Let $f(t) = t^4$. Then, from Eq.(22), $F(s) = 4!/s^5$. So, we have $\frac{d^2F(s)}{ds^2} = \frac{5\times6\times4!}{t^7} = \frac{6!}{t^7} = \mathcal{L}\{t^6\} = \mathcal{L}\{t^2 f(t)\}$ ☺

4. **Real Integration.** If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s} \tag{44}$$

$$\mathcal{L}\left\{\int_{-\infty}^t f(\tau)d\tau\right\} = \frac{1}{s}\int_{-\infty}^0 f(t)dt + \frac{F(s)}{s} \tag{45}$$

In other words, integration in the time domain amounts to a division by s in the frequency domain!

5. **Final value theorem.** Suppose we are interested in determining the *long-term behavior* of a time signal $f(t)$. I.E. we would like to know the limit

$$\lim_{t \to \infty} f(t) \tag{46}$$

When a direct expression for $f(t)$ is available, this is usually easy to do because it simply amounts to enforcing a limit in a known functional form. However, in other cases, it may not be so straightforward. E.g. consider the states of a dynamical system, $\dot{x} = ax$. We need to determine $\lim t \to \infty x(t)$, i.e. what happens to the state after a long period of time? It turns out that the "final value" can be obtained by evaluating an equivalent limit in the frequency domain:

Theorem 1. *Let $f(t)$ and $\dot{f}(t)$ be Laplace transformable. If $\lim_{t\to\infty} f(t)$ exists, then,*

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) \tag{47}$$

The above appears to be a chicken and egg problem – in order to determine the limit, you must know first if it exists – but you cannot know that since you are trying to determine it in the frequency domain... Luckily, there is a good, indirect way of checking if the limit exists – simply check to see if there are no poles of $F(s)$ on the imaginary axis or the right-hand-plane (RHP), i.e. poles with positive real parts. A simple pole at the origin is not a problem.

❣ **Example** Suppose

$$F(s) = \frac{s+2}{s^5 + 2s^4 + 4s^3 + 2s^2 + s} \tag{48}$$

The poles of this function are roots of the polynomial in the denominator: $\{0, -0.7429 \pm j1.5291, -0.2571 \pm j0.5291\}$. Clearly, four of the five poles lie in the left-half plane (LHP) and there is one (simple) pole at the origin. This is shown in Fig.(8), in which the Left and Right-Half planes are also marked. Based on the above, $\lim_{t\to\infty} f(t)$ exists! Moreover, it must equal:

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} \frac{s(s+2)}{s^5 + 2s^4 + 4s^3 + 2s^2 + s} = 2 \tag{49}$$

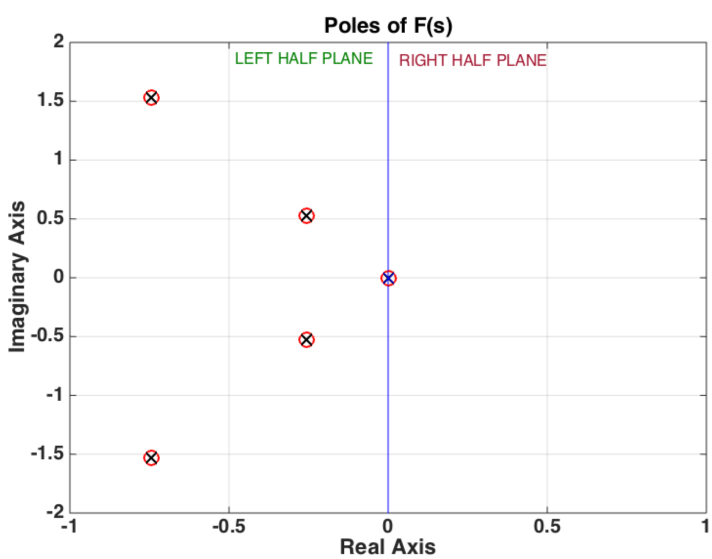


Figure 8: The s-plane. Poles of $F(s)$ shown.

Inverse Laplace Transforms

In order to return to the time-domain, we must invert the Laplace transform:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \tag{50}$$

Typically, Laplace transforms are of the form

$$F(s) = \frac{p(s)}{q(s)} \tag{51}$$

where, $p(s)$ and $q(s)$ are polynomials of s. Clearly, based on the discussion in the previous section, the roots of $p(s)$ are the zeros of $F(s)$ and roots of $q(s)$ are its poles. We will consider two major cases for inverting the LT $F(s)$.

Case I. Only simple poles. In this case, the LT has only simple poles, i.e. poles that are distinct from each other. Therefore, the function $F(s)$ can be factorized as:

$$F(s) = \frac{k(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)} \tag{52}$$

We have, $p_1 \neq p_2 \dots \neq p_n$. However, there is no such restriction on the zeros! Also, in most physical cases, $n > m$. Note that it is not clear what the inverse Laplace transform of $F(s)$ is because it does not match with any of the transforms studied in the previous section (e.g. unit step, monomials, exponential, sinusoids..). The standard way of dealing with them is to propose a *partial-fractions* based decomposition as follows:

$$F(s) \underset{\text{(propose)}}{=} \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \dots \frac{k_n}{s+p_n} \tag{53}$$

The structure on the RHS of the above equation is valid only if $F(s)$ has simple poles. The reason for using partial fractions is that the proposed structure (RHS of Eq.(53)) has terms for which the inverse Laplace transform is known (recall $\mathcal{L}^{-1}\{k/(s+a)\} = ke^{-at}$). Therefore, all we need to do now is find the so-called *residues* $k_1, k_2, \dots, k_n$ of the poles $p_1, p_2, \dots, p_n$, following which

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\} &\underset{\text{Eq.53}}{=} \mathcal{L}^{-1}\left\{\frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \dots \frac{k_n}{s+p_n}\right\} \\
&\underset{\text{linearity}}{=} k_1\mathcal{L}^{-1}\left\{\frac{1}{s+p_1}\right\} + k_2\mathcal{L}^{-1}\left\{\frac{1}{s+p_2}\right\} + \dots + k_n\mathcal{L}^{-1}\left\{\frac{1}{s+p_n}\right\} &\text{(54a)} \\
&= k_1e^{-p_1t} + k_2e^{-p_2t} + \dots + k_ne^{-p_nt} &\text{(54b)} \\
&= \sum_{i=1}^{n} k_ie^{-p_it} &\text{(54c)}
\end{aligned}$$

In the above developments, it doesn't matter if the poles are real, imaginary, or a mixture of both - they must only be distinct.

❢ **Example** Consider the spring mass damper system:

$$m\ddot{x} + b\dot{x} + kx = u(t) \tag{55}$$

We are required to determine $x(t)$, given that $x(0) = x_0$ and $\dot{x}(0) = v_0$, and, $u(t) = 0$ (no control).

The strategy is to solve for $x(t)$ in the Laplace domain and retrieve the time-domain signal by implementing the inverse Laplace transform. We can obtain $\mathcal{L}\{x(t)\} \triangleq X(s)$ using the information given (Eq.(55)):

$$\begin{aligned}
\mathcal{L}\{m\ddot{x} + b\dot{x} + kx\} &= \mathcal{L}\{u(t)\} = 0 \quad \text{(no control)} &\text{(56a)} \\
\Rightarrow \quad m\mathcal{L}\{\ddot{x}\} + b\mathcal{L}\{\dot{x}\} + k\mathcal{L}\{x\} &= 0 &\text{(56b)} \\
\Rightarrow \quad m\underbrace{\left[s^2X(s) - sx_0 - v_0\right]}_{\mathcal{L}\{\ddot{x}\}} + b\underbrace{\left[sX(s) - x_0\right]}_{\mathcal{L}\{\dot{x}\}} + kX(s) &= 0 &\text{(56c)}
\end{aligned}$$

Collecting terms,

$$X(s) = \frac{mx_0 s + (mv_0 + bx_0)}{ms^2 + bs + k} \tag{57}$$

Or,

$$X(s) = \frac{x_0\left[s + \left(\frac{b}{m} + \frac{v_0}{x_0}\right)\right]}{s^2 + \frac{b}{m}s + \frac{k}{m}} \tag{58}$$

The difference between expressions in Eqs.(57) and (58) is that in the latter, the leading terms in the numerator and denominator has unit coefficient. To continue solving the problem, let us assign numbers to the variables: let $b/m = 3$ and $k/m = 2$, such that the denominator is $(s^2 + 3s + 2) \equiv (s+2)(s+1)$. Since the LT of $x(t)$ has two simple poles (-2, -1), the following partial-fraction decomposition can be proposed:

$$X(s) = \frac{x_0\left[s + \left(3 + \frac{v_0}{x_0}\right)\right]}{(s+2)(s+1)} = \frac{k_1}{s+2} + \frac{k_2}{s+1} \tag{59}$$

which gives us

$$k_1(s+1) + k_2(s+2) = x_0\left[s + (3+\chi)\right] \qquad \left(\text{define } \chi = \frac{v_0}{x_0}\right) \tag{60}$$

The unknowns are k_1 and k_2 and the above equation must hold for all possible values of s. There are many ways of obtaining k_1 and k_2.. the most obvious one is to match the coefficient of each power-of-s on the two sides of the equation, which would give us:

$$\begin{aligned} k_1 + k_2 &= x_0 \quad (\text{coefficient of } s^1) \\ k_1 + 2k_2 &= x_0(3+\chi) \quad (\text{coefficient of } s^0) \end{aligned}$$

The above approach is not very attractive because it leads to two coupled equations which must be solved simultaneously. An easier way is to evaluate Eq.(60) at carefully selected values of s, thereby allowing us to determine each undetermined variable one at a time.. consider the following..

evaluate Eq.(60) at $s = -1$: $k_1(-1+1) + k_2(-1+2) = x_0[-1 + (3+\chi)]$

Or, $k_2 = x_0(2+\chi)$ (61a)

evaluate Eq.(60) at $s = -2$: $k_1(-2+1) + k_2(-2+2) = x_0[-2 + (3+\chi)]$

Or, $k_1 = -x_0(1+\chi)$ (61b)

Eq.(59) now becomes

$$X(s) = \frac{x_0(2+\chi)}{s+1} - \frac{x_0(1+\chi)}{s+2} \tag{62}$$

This can be easily inverted to give:

$$x(t) = \mathcal{L}^{-1}\left\{\frac{x_0(2+\chi)}{s+1} - \frac{x_0(1+\chi)}{s+2}\right\} = x_0\left[\left(2 + \frac{v_0}{x_0}\right)e^{-t} - \left(1 + \frac{v_0}{x_0}\right)e^{-2t}\right] \tag{63}$$

Two ways of determining $\dot{x}(t) = v(t)$:

- (time) differentiate $x(t)$ just obtained:

$$v(t) = x_0\left[2\left(1 + \frac{v_0}{x_0}\right)e^{-2t} - \left(2 + \frac{v_0}{x_0}\right)e^{-t}\right] \tag{64}$$

- Go the Laplace route: we know that $V(s) = \mathcal{L}\{v(t)\} = \mathcal{L}\{\dot{x}(t)\} = sX(s) - x_0$. I.E.,

$$V(s) = sX(s) - x_0 = x_0\left[(2+\chi)\frac{s}{s+1} - (1+\chi)\frac{s}{s+2} - 1\right]$$

$$= x_0\left[(2+\chi)\frac{s+1-1}{s+1} - (1+\chi)\frac{s+2-2}{s+2} - 1\right]$$

$$= x_0\left[(2+\chi)\left(1 - \frac{1}{s+1}\right) - (1+\chi)\left(1 - \frac{2}{s+2}\right) - 1\right]$$
$$= x_0\left[-\frac{2+\chi}{s+1} + \frac{2(1+\chi)}{s+2}\right]$$

Thus,
$$v(t) = \mathcal{L}^{-1}\left\{-\frac{(2+\chi)x_0}{s+1} + \frac{2(1+\chi)x_0}{s+2}\right\} = x_0\left[2\left(1+\frac{v_0}{x_0}\right)e^{-2t} - \left(2+\frac{v_0}{x_0}\right)e^{-t}\right] \tag{65}$$

Case (1.a). Dealing with complex poles. We are concerned here with a special case of case I: assuming that the poles are simple, but some of them are complex. The procedure described in the above case can still be used here (i.e. partial fraction decomposition followed by reconstruction of the time-domain signal in terms of exponentials), but can entail some nasty complex-algebra. There is another, simpler way of dealing with complex poles described below via an example..

❢ **Example** Consider the following LT:

$$\mathcal{L}\{f(t)\} = \frac{1}{s^3 + 7s^2 + 17s + 15} \tag{66}$$

The poles of the above LT are: $(-3,\ -2 \pm j)$. The procedure described in case I would have us factorize $F(s)$ as:

$$F(s) = \frac{1}{(s+2+j)(s+2-j)(s+3)} \tag{67}$$

which is perfectly alright, except that as mentioned above, will require some complex-algebra down the line. This can be avoided by *combining* the complex poles into a single term:

$$F(s) = \frac{1}{(s^2+4s+5)(s+3)} \tag{68}$$

The motivation behind this step is that quadratic polynomials with complex roots can be expressed as: $(s+a)^2 + \omega^2$... for instance..in the above example,

$$\frac{1}{s^2+4s+5} = \frac{1}{(s^2+4s+4+1)} = \frac{1}{(s+2)^2+1} \tag{69}$$

In fact, its roots are: $s = -a \pm j\omega$. The important thing is, we <u>know</u> the inverse Laplace transform of the function in Eq.(69).. its $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} = e^{-2t}\sin t$ (combining Eqs.(24) and (29)) .

We now proceed with partial fraction decomposition of $F(s)$, factorized as in Eq.(68)..

$$F(s) \underset{\text{propose}}{=} \frac{k_1 s + k_2}{s^2+4s+5} + \frac{k_3}{s+3} \tag{70}$$

Note that the proposal in the above equation is slightly different from the proposal of Eq.(53).. this is because the first term in the partial-fraction is a quadratic (i.e. order 2).. which requires us to have a polynomial of order 1 in the proposed residue of that term. Comparing numerators of Eqs.(66) and (70) we get:

$$(k_1 s + k_2)(s+3) + k_3(s^2+4s+5) = 1 \equiv 0.s^2 + 0.s + 1 \tag{71}$$

As mentioned before, you may proceed to find $k_1 - k_3$ by comparing coefficients of the various powers of s on either side of the equation – but this approach is time-consuming. We therefore evaluate the above equation at the following carefully chosen values of s (since the it must hold $\forall\ s$):

$$\begin{aligned} \text{at } s = -3: \quad (k_1 + k_2).0 + 2k_3 &= 1 \\ \text{Or,} \quad k_3 &= 1/2 \end{aligned} \tag{72a}$$

$$\begin{aligned} \text{at } s = 0: \quad 3(k_1.0 + k_2) + 5k_3 &= 1 \\ \text{Or, } k_2 &= (1 - 5k_3)/3 = -1/2 \end{aligned} \tag{72b}$$

$$\begin{aligned} \text{at } s = -2: \quad (-2k_1 + k_2) + k_3 &= 1 \\ \text{Or, } k_2 &= -(1 - k_3 - k_2)/2 = -1/2 \end{aligned} \tag{72c}$$

So we get

$$F(s) = \frac{1}{2}\left[\frac{1}{s+3} - \underbrace{\frac{s+1}{(s+2)^2+1}}_{T_2}\right] \tag{73}$$

The second term above (T_2) needs to be manipulated a little more..

$$\begin{aligned} T2 &= \frac{s+1}{(s+2)^2+1} = \frac{(s+2)-1}{(s+2)^2+1} \\ &= \underbrace{\frac{(s+2)}{(s+2)^2+1}}_{T_{21}} - \underbrace{\frac{1}{(s+2)^2+1}}_{T_{22}} \end{aligned} \tag{74}$$

Term T_{22} is easy:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} = e^{-2t}\sin t \tag{75}$$

For T_{21}, note the following: Let $G(s) = \frac{\omega}{s^2+\omega^2}$. Then, $\mathcal{L}^{-1}\{G(s)\} = g(t) = \sin\omega t$. Also, from the (real) differentiation rule (Eq.(37)),

$$\mathcal{L}^{-1}\{sG(s)\} = \frac{dg(t)}{dt} + \mathcal{L}^{-1}\{g(0)\} \tag{76}$$

Here, $g(t) = \sin\omega t$, so $dg/dt = \omega\cos\omega t$ and $g(0) = 0$. Therefore,

$$\mathcal{L}^{-1}\left\{\frac{\omega s}{s^2+\omega^2}\right\} = \omega\cos\omega t \tag{77}$$

Adding in the damping effect:

$$\mathcal{L}^{-1}\left\{\frac{\omega(s+a)}{(s+a)^2+\omega^2}\right\} = \omega e^{-at}\cos\omega t \tag{78}$$

Term T_{21} is now done because you identify $\omega = 1$ and $a = 2$ to get:

$$\mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2+1}\right\} = e^{-2t}\cos t \tag{79}$$

Combining Eqs.(79), (75), (74) and (73), we finally get

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}\left[e^{-3t} + e^{-2t}(\sin t - \cos t)\right] \tag{80}$$

Case II. Non simple poles. In this case, we will consider an example with repeated poles, i.e. poles of order > 1. Consider the example:

❢ **Example**

$$F(s) = \frac{s^2+2s+3}{(s+1)^3} \tag{81}$$

There are three repeated poles at $s = -1$. In this case, the proposed partial-fraction structure looks like:

$$F(s) \underset{\text{propose}}{=} \frac{k_1}{(s+1)} + \frac{k_2}{(s+1)^2} + \frac{k_3}{(s+1)^3} \tag{82}$$

Comparing the numerators again, we get:

$$k_1(s+1)^2 + k_2(s+1) + k_3 = s^2 + 2s + 3 \qquad \forall\ s \tag{83}$$

As mentioned before, the easy way to determine k_1 - k_3 is to evaluate Eq.(83) at carefully selected points.. At $s = -1$, we get

$$k_3 = 2 \tag{84}$$

Now there is a problem.. because all other evaluations of Eq.(83) will contain both k_1 and k_2. So, we introduce another trick.. since Eq.(83) holds for all s, its derivatives (wrt s) must also hold for all s! So, differentiating this equation **once** with respect to s we get:

$$2k_1(s+1) + k_2 = 2s+2 \tag{85}$$

This equation can now be evaluated at $s = -1$ to get $k_2 = 0$. Differentiate one more time:

$$2k_1 = 2 \tag{86}$$

which immediately gives $k_1 = 1$! So, $F(s)$ of Eq.(81) is equivalent to:

$$F(s) = \frac{1}{s+1} + \underbrace{\frac{2}{(s+1)^3}}_{T2} \tag{87}$$

For $T2$, recall that $\mathcal{L}^{-1}\{2!/s^3\} = t^2$. Thus from the damping rule, $\mathcal{L}^{-1}\{2!/(s+1)^3\} = e^{-t}t^2$. We finally get:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = e^{-t} + e^{-t}t^2 = e^{-t}(1+t^2) \tag{88}$$

⊛ The above three cases describe how you must handle the inverse Laplace transform after you determine the nature of the poles. In case that you a get "mixed" types of poles.. e.g. combination of simple, complex, repeated all in one... then you must combine the rules from the individual cases described above to come up with the appropriate proposal for the partial fraction expansion.