EML 4312: Control of Mechanical Engineering Systems

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Transfer Functions

Definition 1. *For a linear system, the transfer function is the ratio of Laplace transform of the output signal to the input signal, assuming all initial conditions are set to zero.*

Consider the two-block system shown in Fig.(1). We consider the "system" to be defined by the dotted-box. Clearly, the input signal is $r(t)$ and the output is $y(t)$. Following our usual notation,

$$\mathcal{L}\{y(t)\} = Y(s); \;\; \mathcal{L}\{r(t)\} = R(s) \tag{1}$$

Thus following Def.(1), we have

$$G_{\text{sys}}(s) \triangleq \frac{Y(s)}{R(s)} \tag{2}$$

where, $G_{\text{sys}}(s)$ denotes the "transfer function", as clearly, it is a function of the variable s.

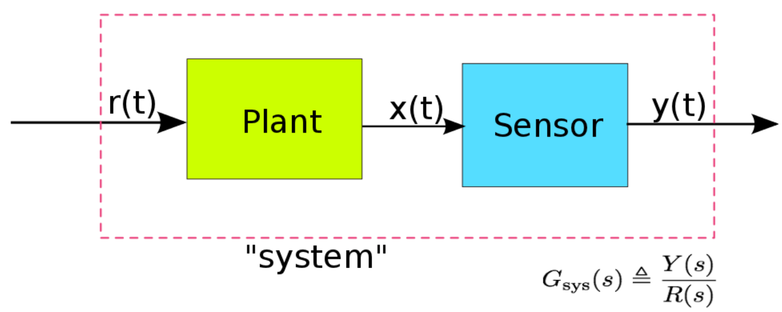


Figure 1: Illustration of the "transfer function": input-output relationship in the frequency domain

- The definition given in Eq.(1) can also be applied to individual blocks in Fig.(1). For example, the transfer function of the sensor is:

$$G_{\text{sensor}}(s) = \frac{Y(s)}{X(s)} \tag{3}$$

  since the input to the sensing block is $x(t)$ and its output is $y(t)$. Similarly, the transfer function of the plant is

$$G_{\text{plant}}(s) = \frac{X(s)}{R(s)} \tag{4}$$

- Note that the transfer function describes only the input-output behavior. All other signals and other "details of the internal composition of the system" is "washed out". For instance, the signal $x(t)$ is hidden in the transfer function G_{sys}.

❣ **Example** Consider our spring-mass-damper system

$$m\ddot{x} + b\dot{x} + kx = u(t) \tag{5}$$

We consider two separate cases of output:

$$y(t) = x(t) \tag{6a}$$
$$y(t) = \dot{x}(t) \tag{6b}$$

In the first case, we are measuring the position of the block while in the second, the speed. In terms of the block diagram shown in Fig.(1), Eq.(5) represents the *plant* while Eqs.(6) represent two different cases of the sensor. Let us consider the two cases separately:

(a) $y(t) = x(t)$. We first take the Laplace transform of the dynamic system (Eq.(5)) to get (setting all initial conditions to zero):

$$ms^2X(s) + bsX(s) + kX(s) = U(s) \tag{7}$$

where, $X(s) = \mathcal{L}\{x(t)\}$ and $U(s) = \mathcal{L}\{u(t)\}$. Therefore,

$$\frac{X(s)}{U(s)} = \frac{1}{ms^2 + bs + k} = G_{\text{plant}}(s)! \tag{8}$$

From Eq.(6a),

$$Y(s) = X(s) \tag{9}$$

thus,

$$\frac{Y(s)}{X(s)} = 1 = G_{\text{sensor}}(s)! \tag{10}$$

Thus, the overall system transfer function is: $G_{\text{sys}}(s) = \frac{Y(s)}{U(s)}$:

$$G_{\text{sys}}(s) = \frac{Y(s)}{U(s)} \tag{11a}$$
$$= \underbrace{\frac{Y(s)}{X(s)}}_{G_{\text{sensor}}(s)} \cdot \underbrace{\frac{X(s)}{U(s)}}_{G_{\text{plant}}(s)} \tag{11b}$$
$$\underset{\text{Eqs.8,9}}{=} 1 \cdot \frac{1}{ms^2 + bs + k} = \frac{1}{ms^2 + bs + k} \tag{11c}$$

The most important thing to note in the above developments is how we "cascaded" the individual block transfer functions to obtain the overall system transfer function (Eq.(11b)):

$$G_{\text{sys}}(s) = G_{\text{sensor}}(s) \times G_{\text{plant}}(s) \tag{12}$$

(b) $y(t) = \dot{x}(t)$. This case is similar, except that Eq.(9) must be replaced with:

$$Y(s) = sX(s) \tag{13}$$

so that

$$\frac{Y(s)}{X(s)} = \frac{1}{s} \tag{14}$$

The rest of the developments are the same, giving us

$$G_{\text{sys}}(s) = \frac{Y(s)}{U(s)} \tag{15a}$$
$$= \underbrace{\frac{Y(s)}{X(s)}}_{G_{\text{sensor}}(s)} \cdot \underbrace{\frac{X(s)}{U(s)}}_{G_{\text{plant}}(s)} \tag{15b}$$
$$\underset{\text{Eqs.8,13}}{=} \frac{1}{s} \cdot \frac{1}{ms^2 + bs + k} = \frac{1}{s(ms^2 + bs + k)} \tag{15c}$$

- **Zeros, Poles and Gain.** Since a transfer function is nothing but the ratio of two Laplace transforms, we can talk about their zeros, poles and gain. A general transfer function can be written as:

$$G(s) = \frac{P(s)}{Q(s)} \tag{16}$$

  - Clearly, the numerator $P(s)$ can be factorized as:

$$P(s) = k\,(s - z_1)(s - z_2)\ldots(s - z_m) = k\prod_{i=1}^{m}(s - z_i) \tag{17}$$

    such that the *zeros* of the transfer function $G(s)$ are $\{z_1, z_2, \ldots, z_m\}$.
  - Similarly, the denominator can be factorized as:

$$Q(s) = (s - p_1)(s - p_2)\ldots(s - p_n) = \prod_{i=1}^{n}(s - p_i) \tag{18}$$

    such that the *poles* of the transfer function $G(s)$ are $\{p_1, p_2, \ldots, p_n\}$. Because of the significance of the polynomial $Q(s)$ (we will soon unravel this), it is called the *characteristic polynomial* of the transfer function, and the equation $Q(s) = 0$ is called the *characteristic equation*. Clearly, the roots of the characteristic polynomial are the poles of the transfer function.
  - In Eq.(17), the constant k is called the *gain* of the transfer function.

- Let us consider another example: the op-amp:

❣ **Example** Recall that the op-amp is the modern day realization of Black's feedback amplifier. Its generic structure is given in Fig.(2(a)). The system is designed to (ideally) have infinite impedance, such that $i_1 = i_2 = 0$, and also, $v_1 = v_i = v_2$. There are two places to hook up the input signal to: v_i and v_2. The node v_i is called the *inverting* node, and the latter is the *non-inverting node*. In the so-called "inverting mode" the circuit is realized as shown in Fig.(2(b)), where the input signal is hooked onto the node v_i while the node v_2 is grounded.

At junction N1, the sum of currents must be zero (Kirchoff's first law), so that:

$$-\frac{v_1 - v_i}{R_1} + \underbrace{i_1}_{=0\,(\infty\text{ impedance})} + \frac{v_o - v_1}{R_2} = 0 \tag{19}$$

Now, under ideal conditions, $v_1 = 0$ (since $v_1 = v_2$ and v_2 is grounded). So, the above equation gives us that

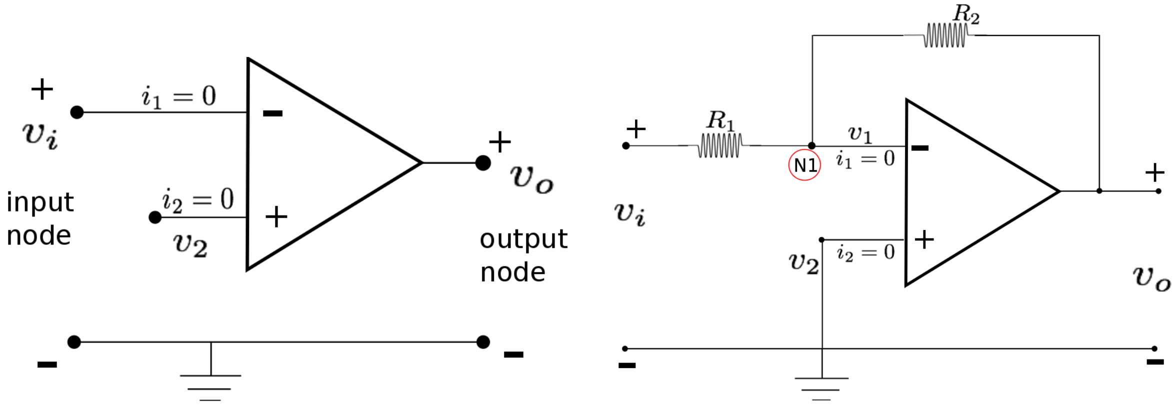
$$v_o = -\frac{R_2}{R_1}v_i \tag{20}$$

The negative sign in the above equation reaffirms the "inversion", while the ratio of R_2 to R_1 is the magnitude of amplification. If we take the Laplace transform,

$$V_o(s) = -\frac{R_2}{R_1}V_i(s) \tag{21}$$

such that the transfer function is

$$G(s) = \frac{V_o(s)}{V_i(s)} = -\frac{R_2}{R_1} \quad \text{(a constant)} \tag{22}$$



(a) Generic Structure (b) Inverting Configuration

Figure 2: An Operational Amplifier (Op-Amp)

❣ **Example (DC Motor)**. Next we consider a more elaborate example: the DC motor, which is an electromechanical device. It uses electrical energy to generate mechanical effect. In terms of a transfer function, this means that the input signal is of electrical origin while the output signal is of the mechanical type. There are two main components of a DC motor: (i) the *field*, which is a fixed wound coil, and (ii) the *armature*, which is a rotating wound coil. The mechanical load is attached to the armature, as shown in Fig.(3).

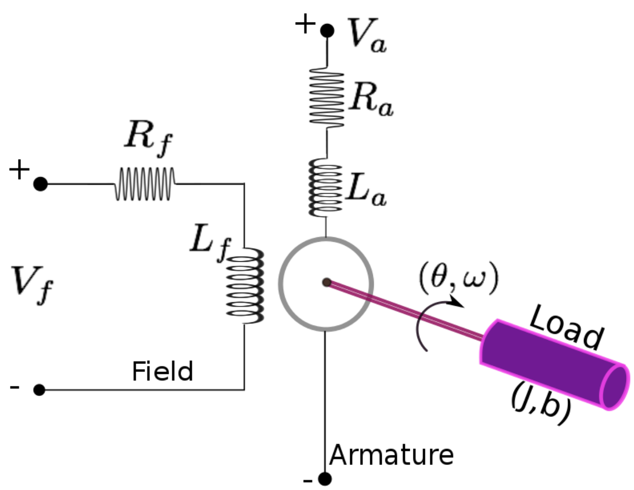


Figure 3: DC motor layout

The input is *either* the field voltage (V_f) *or* the armature voltage (V_a). In the former case, the motor is said to be *field controlled*, while in the latter, it is called *armature controlled*. In the field controlled mode, the armature voltage and current are held constant, while in the armature mode, the field voltage and current are constant. The output is either the angular position, θ, or the speed, ω. The motor generates a torque, T_m (motor torque) that drives the shaft connecting to the load. The generated motor torque is given by

$$T_m = K_1 K_f i_f i_a \tag{23}$$

where, K_1 and K_f are constants and i_f is the field current and i_a is the armature current. Clearly, to ensure linearity, one of the two currents must be held constant! This is the main idea behind operating the motor in either the field mode or the armature mode. Let us consider the two cases separately:

– **Field Controlled DC Motor.** In this case, the armature voltage and current are held constant.

Thus, in Eq.(23), we define $K_m = K_1 K_f i_a$, to give

$$T_m = K_m i_f \tag{24}$$

Taking Laplace transform, we get

$$\mathtt{T}_m(s) = K_m I_f(s) \tag{25}$$

where, $\mathtt{T}_m(s) = \mathcal{L}\{T_m\}$.

Note that the input signal is V_f and not I_f.. so we must get the electrical relationship between these quantities. The field coil is isolated from the load, so that it is a simple resistor-inductor series connection with a voltage drop of v_f, as shown below in Fig.(4).

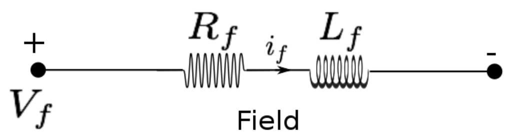


Figure 4: Field Coil

The dynamics of the above circuit is:

$$v_f = R_f i_f + L_f \frac{di_f}{dt} \tag{26}$$

Taking Laplace transform, we get

$$V_f(s) = R_f I_f(s) + L_f s I_f(s) \tag{27}$$

or,

$$V_f(s) = (L_f s + R_f) I_f(s) \tag{28}$$

Now, the load is driven by the *load torque*, which is the motor torque *minus* disturbances/losses, denoted by T_d:

$$T_l = T_m - T_d \tag{29}$$

or, in the Laplace domain,

$$\mathtt{T}_l(s) = \mathtt{T}_m(s) - \mathtt{T}_d(s) \tag{30}$$

The relationship between the load torque and the load is given by the following mechanical equation (describing rotational dynamics):

$$T_l = J\ddot{\theta} + b\dot{\theta} \tag{31}$$

which is clearly a second order system. Taking the Laplace transform (and as always, assuming zero initial conditions), we get

$$\mathtt{T}_l(s) = Js^2\Theta(s) + bs\Theta(s) \tag{32}$$

where, $\Theta(s) = \mathcal{L}\{\theta\}$. Or,

$$\mathtt{T}_l(s) = (Js^2 + bs)\Theta(s) \tag{33}$$

It is time to build the overall transfer function for the field controlled DC motor! We have the following flow of variables:

(a.) The input is the field voltage: V_f. The electrical relationship given in Eq.(26) generates the field current i_f.

(b.) The field current couples with the armature to generate the motor torque, T_m (Eq.(24)). This relationship captures the electro-mechanical conversion.

(c.) The motor torque is transmitted to the load after incurring losses, T_d, given by Eq.(29), resulting in the load torque, T_l.

(d.) The load torque (T_l) drives the load, given by Eq.(31). This is a purely mechanical relationship.

Overall, we have:

$$v_f \underset{\text{Eq.26}}{\longrightarrow} i_f \underset{\text{Eq.24}}{\longrightarrow} T_m \underset{\text{Eq.29}}{\longrightarrow} T_l \underset{\text{Eq.31}}{\longrightarrow} (\theta, \omega) \tag{34}$$

where, $\omega = \dot{\theta}$. The final output is either θ or ω, depending on the application. For example, if we have a disc reading system, we would like to control the angular location of the "reader", so that the output variable is θ. On the other hand, the output for a car-cruise control system is the speed, $\omega = \dot{\theta}$. For our current analysis, let the final output be the angular position, θ.

Note that Eq.(34) is simply a sequence of input and output signals. The "overall" system input is v_f and the "overall" system output is θ. In between, we have several components (or blocks), each with their own input and output signals. Therefore it makes sense to draw a block diagram to depict the above flow: this is shown in Fig.(5)

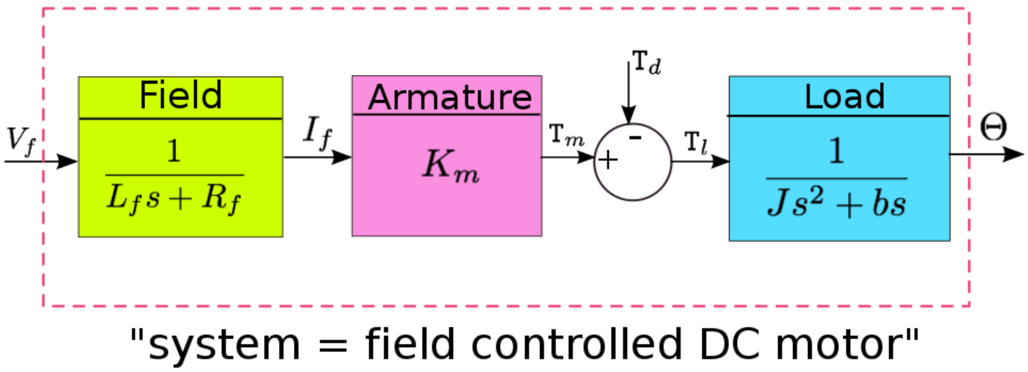


Figure 5: DC motor block diagram: Field Controlled Mode

The overall transfer function for the field controlled DC motor is (by definition):

$$G_{\text{DC-F}}(s) = \frac{\Theta(s)}{V_f(s)} \tag{35}$$

Under ideal operating conditions, $T_d = 0$ (i.e. zero losses) and we have $T_l = T_m$. Given the sequence in Eq.(34), it is easy to construct the above transfer function as follows:

$$G_{\text{DC-F}}(s) = \underbrace{\frac{\Theta(s)}{T_l(s)}}_{G_{\text{Load}}(s)} \cdot \underbrace{\frac{T_l(s)}{T_m(s)}}_{=1\,!!} \cdot \underbrace{\frac{T_m(s)}{I_f(s)}}_{G_{\text{Arm}}(s)} \cdot \underbrace{\frac{I_f(s)}{V_f(s)}}_{G_{\text{Field}}(s)} \tag{36}$$

The above "cascading" allows us to construct the <u>desired</u> transfer-function ($G_{\text{DC-F}}(s)$) in terms of <u>simpler, known, single-block</u> transfer functions: $G_{\text{Load}}(s)$, $G_{\text{Arm}}(s)$ and $G_{\text{Field}}(s)$. We have:

$$G_{\text{Load}}(s) = \frac{\Theta(s)}{T_l(s)} \underset{\text{Eq.33}}{=} \frac{1}{Js^2 + bs} \tag{37a}$$

$$G_{\text{Arm}}(s) = \frac{T_m(s)}{I_f(s)} \underset{\text{Eq.30}}{=} K_m \tag{37b}$$

$$G_{\text{Field}}(s) = \frac{I_f(s)}{V_f(s)} \underset{\text{Eq.28}}{=} \frac{1}{L_f s + R_f} \tag{37c}$$

Putting all of the above block transfer functions together into Eq.(36), we get:

$$G_{\text{DC-F}}(s) = G_{\text{Field}}(s) \times G_{\text{Arm}}(s) \times G_{\text{Load}}(s) = \frac{K_m}{s(Js+b)(L_f s + R_f)} \tag{38}$$

Keep in mind that the above assumes that the disturbance torque (T_d) is absent.

♣ The above analysis carries an important lesson – *the transfer function of a "series connection" (i.e. open loop!!) of blocks is the product of the transfer functions of the individual blocks!!*

Removing the disturbance torque from Fig.(5), the illustration is given in Fig.(6). We have, the "open-loop transfer function":

$$G_{\text{DC-F}}(s) = G_{\text{ol}}(s) = G_{\text{Field}}(s) \times G_{\text{Arm}}(s) \times G_{\text{Load}}(s) \tag{39}$$

where, $G_{\text{ol}}(s)$ stands for the transfer function of the "open-loop" connection of the three blocks (field, armature and load).

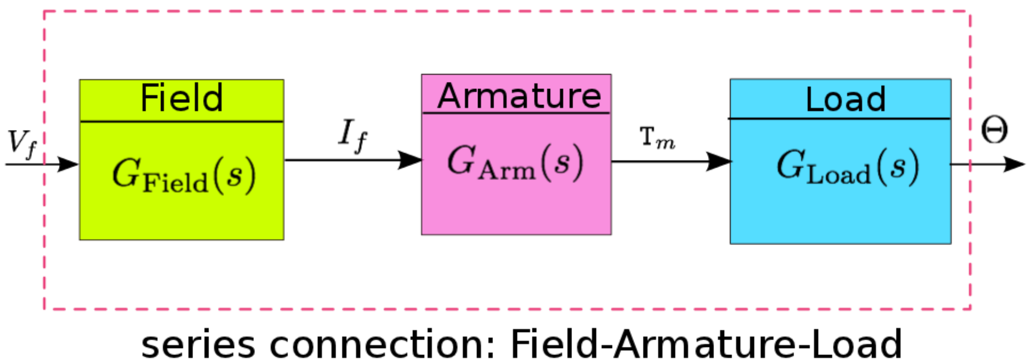


Figure 6: Series Connection of Blocks (i.e. Open-Loop Connection)

– **Armature Controlled DC Motor.** In the armature controlled mode of the DC motor, the field current is held constant and the armature voltage is used as the control input. Therefore, in Eq.(23), we define the constant as $K_m = K_1 K_f i_f$, such that

$$T_m = K_m i_a \tag{40}$$

So that in the frequency domain,

$$\mathtt{T}_m(s) = K_m I_a(s) \tag{41}$$

The relationship between the armature voltage (v_a) and armature current is somewhat more complex than the field-controlled case. This is because of the load connected to the armature, which when rotating, creates a "back emf", v_b that is proportional to the rate of rotation (ω). This is shown in Fig.(7).

Due to the additional voltage drop, the circuit equation is

$$v_a - v_b = R_a i_a + L_a \frac{di_a}{dt} \tag{42}$$

where, the back emf is

$$v_b = K_b \omega = K_b \dot{\theta} \tag{43}$$

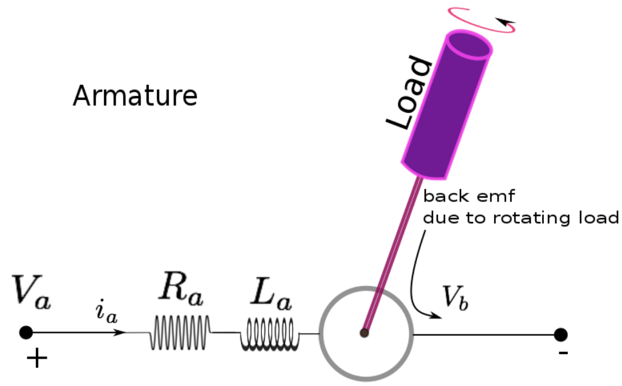


Figure 7: Armature Coil: Back EMF

Taking Laplace transform of Eq.(42), we get

$$V_a(s) - V_b(s) = (R_a + L_a s)I_a(s) \tag{44}$$

Also,

$$V_b(s) = K_b\Omega(s) = K_b s\Theta(s) \tag{45}$$

where, $\Omega(s) = \mathcal{L}\{\omega(t)\}$. Thus, combining Eqs.(44) and (45) we get

$$I_a(s) = \frac{V_a(s) - sK_b\Theta(s)}{L_a s + R_a} \tag{46}$$

The relationship between motor torque and load torque is identical as for the field controlled case (Eq.(29)). So is the mechanical equation between load torque and output variable, θ (Eq.(31)). Therefore, following the same cascading approach as in the previous case, we get a relationship between various variables as shown in Fig.(8) below:

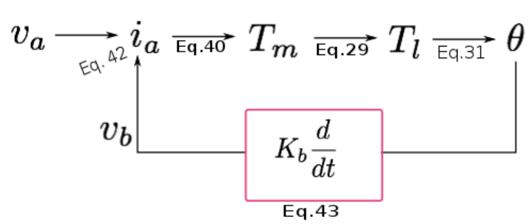


Figure 8: Variable-flow: Armature Controlled DC Motor

Note that the back-emf (v_b) appears as a negative feedback term since it is dependent on the output variable (θ). The transfer function between the overall input v_a and output θ can be constructed in the same way as before:

$$G_{\text{DC-A}}(s) = \frac{\Theta(s)}{V_a(s)} = \underbrace{\frac{\Theta(s)}{\mathtt{T}_l(s)}}_{G_{\text{Load}}(s)} \cdot \underbrace{\frac{\mathtt{T}_l(s)}{\mathtt{T}_m(s)}}_{=1\,!!} \cdot \underbrace{\frac{\mathtt{T}_m(s)}{I_a(s)}}_{G_{\text{Field}}(s)} \cdot \underbrace{\frac{I_a(s)}{V_a(s)}}_{G_{\text{Arm}}(s)} \tag{47}$$

We have the individual terms as follows:

$$G_{\text{Load}}(s) = \frac{\Theta(s)}{\mathrm{T}_l(s)} \underset{\text{Eq.33}}{=} \frac{1}{Js^2 + bs} \tag{48a}$$

$$G_{\text{Field}}(s) = \frac{\mathrm{T}_m(s)}{I_a(s)} \underset{\text{Eq.41}}{=} K_m \tag{48b}$$

$$G_{\text{Arm}}(s) = \frac{I_a(s)}{V_a(s)} \underset{\text{Eq.46}}{=} \frac{1 - sK_b\frac{\Theta(s)}{V_a(s)}}{L_a s + R_a} \tag{48c}$$

Looking at Eq.(48c) closer, notice that $\frac{\Theta(s)}{V_a(s)} = G_{\text{DC-A}}(s)$. Thus,

$$G_{\text{Arm}}(s) = \frac{1 - sK_b G_{\text{DC-A}}(s)}{L_a s + R_a} \tag{49}$$

Therefore, substituting Eqs.(49), (48b) and (48a) into Eq.(47), we get

$$G_{\text{DC-A}}(s) = \frac{1}{Js^2 + bs} \cdot 1 \cdot K_m \cdot \frac{(1 - sK_b G_{\text{DC-A}}(s))}{L_a s + R_a} \tag{50}$$

which solves out to give:

$$\boxed{G_{\text{DC-A}}(s) = \frac{K_m}{s[(Js+b)(L_a s + R_a) + K_b K_m]}} \tag{51}$$

The block diagram representation of the armature controlled DC motor is given in Fig.(9). This figure is equivalent to Fig.(8), except that Fig.(8) is in the time-domain while Fig.(9) is in the frequency domain. Also note the equivalence: $K_b \frac{d}{dt} \equiv K_b s$.

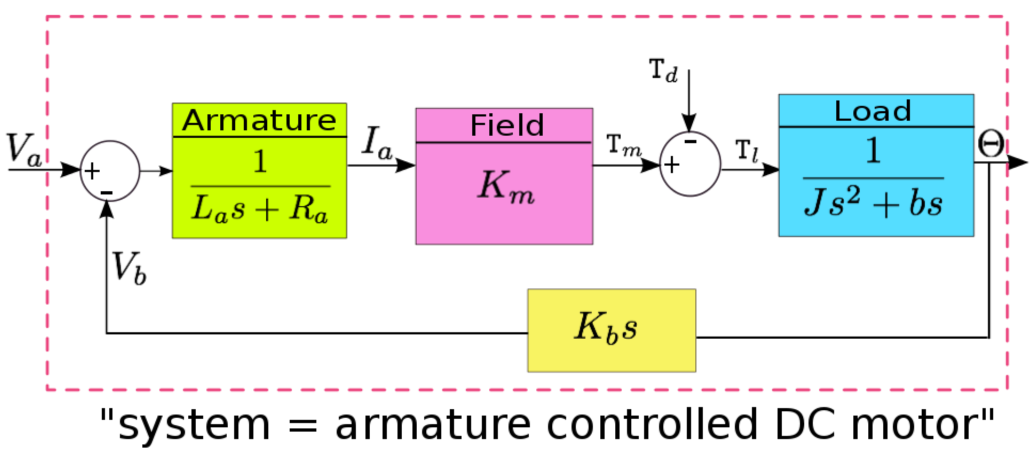
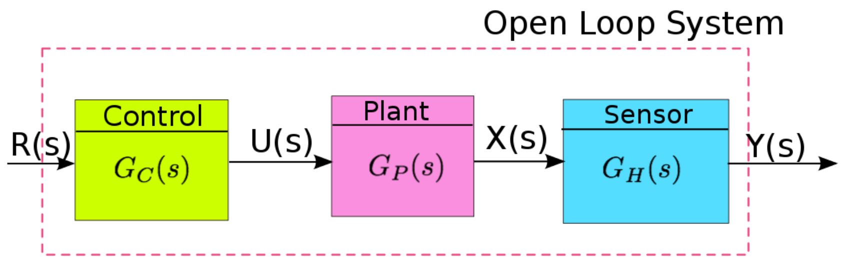


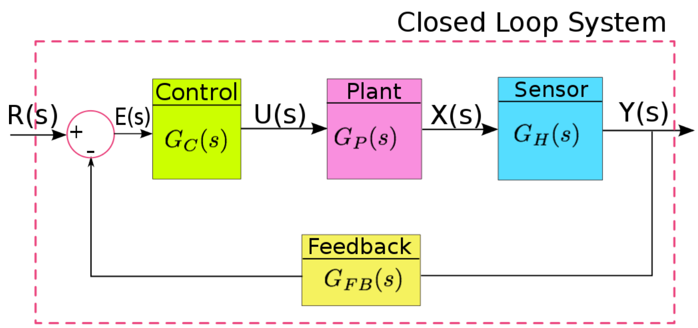
Figure 9: Block Representation of the Armature Controlled DC Motor

As you can see, the transfer function of the feedback structure is more complicated than the open-loop structure. The system transfer function is no longer simply a product of the transfer functions of individual blocks.

- We found out above through the examples related to the DC motor that the transfer function of an open-loop block structure is simply the product of transfer functions of individual blocks. However, things are more complicated in the feedback case. To see the connection between the two in a general setting, consider generic open and closed loop models, shown in Figs.(10). The transfer function for each individual block is shown.



(a) A Generic Open-Loop System



(b) A Generic Closed-Loop System

Figure 10: Transfer Functions of Open and Closed Loop Systems

  - The open loop analysis is easy.. We have

$$G_{\text{ol}}(s) = \frac{Y(s)}{R(s)} \tag{52a}$$

$$= \frac{Y(s)}{X(s)} \cdot \frac{X(s)}{U(s)} \cdot \frac{U(s)}{R(s)} \tag{52b}$$

$$= G_H(s)G_P(s)G_C(s) \tag{52c}$$

    Simply reaffirming the fact that the overall transfer function of the open-loop system is the product of the individual transfer functions.

  - For the closed loop analysis (Fig.(10(b))), we have similar developments:

$$G_{\text{cl}}(s) = \frac{Y(s)}{R(s)} \tag{53}$$

    which can be written as:

$$\frac{Y(s)}{R(s)} = \frac{Y(s)}{X(s)} \cdot \frac{X(s)}{U(s)} \cdot \frac{U(s)}{E(s)} \cdot \frac{E(s)}{R(s)} \tag{54}$$

From Fig.(10(b)), we get $E(s) = R(s) - G_{FB}(s)Y(s)$ (output of the signal adding junction):

$$\frac{Y(s)}{R(s)} = G_H(s)G_P(s)G_C(s)\frac{R(s) - G_{FB}(s)Y(s)}{R(s)} \quad (55a)$$

$$= G_H(s)G_P(s)G_C(s)\left(1 - G_{FB}(s)\frac{Y(s)}{R(s)}\right) \quad (55b)$$

(55c)

Rearranging,

$$\frac{Y(s)}{R(s)}(1 + G_H(s)G_P(s)G_C(s)G_{FB}(s)) = G_H(s)G_P(s)G_C(s) \quad (56)$$

Using Eq.(52c), we get

$$G_{\text{cl}}(s)(1 + G_{\text{ol}}(s)G_{FB}(s)) = G_{\text{ol}}(s) \quad (57)$$

Or,

$$\boxed{G_{\text{cl}}(s) = \frac{G_{\text{ol}}(s)}{1 + G_{FB}(s)G_{\text{ol}}(s)}} \quad (58)$$

Which is a very useful result!

❣ **Example** Going back to out armature controlled DC motor in the ideal case of $T_l = T_m$ (see Fig.(11)), we have the following relationships:

$$G_{\text{ol}}(s) = \frac{\Theta_{\text{ol}}(s)}{V_a(s)} = G_{\text{Arm}}(s)G_{\text{Field}}(s)G_{\text{Load}}(s) = \frac{K_m}{s(Js + b)(L_a s + R_a)} \quad (59)$$

which is the open-loop transfer function, i.e. the transfer function without the feedback signal V_b. When the back-emf (V_b) is active, we have the feedback-block transfer function: $G_{FB}(s) = K_b s$ (compare with Fig.(10(b))). Therefore, using the result of Eq.(58), we must have

$$G_{\text{cl}}(s) = \frac{\Theta_{\text{cl}}(s)}{V_a(s)} = \frac{G_{\text{ol}}(s)}{1 + G_{FB}(s)G_{\text{ol}}(s)} = \frac{\frac{K_m}{s(Js+b)(L_a s+R_a)}}{1 + sK_b\frac{K_m}{s(Js+b)(L_a s+R_a)}} \quad (60)$$

which cleans out to give:

$$G_{\text{cl}}(s) = \frac{K_m}{s[(Js + b)(L_a s + R_a) + K_b K_m]} \quad (61)$$

which is identical to the expression derived above for $G_{DC\text{-}A}(s)$ (Eq.(51)), as it should be!!

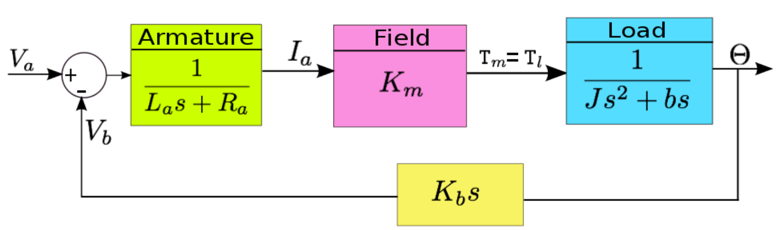


Figure 11: Ideal Armature Controlled DC Motor