Homework 2

Problem 1

Using Matlab, plot the vibration response of a simple spring-mass system initially displaced by 0.4 mm with velocity 1 mm/s. The mass of the system is 100kg with spring stiffness given by 225 N/m.

EOM $\ddot{x} + \frac{k}{m}x = 0$

$$x(0) = 0.4mm$$

$$\dot{x}(0) = 1\frac{mm}{s}$$

$$k = 225\ N/m$$

$$m = 100\ kg$$

Trial solution

$$x = Asin(\omega_n t + \phi)$$

$$\dot{x} = A\omega_n cos(\omega_n t + \phi)$$

$$\ddot{x} = -A\omega_n^2 sin(\omega_n t + \phi)$$

Plug in to EOM

$$Asin(\omega_n t + \phi)\left[-\omega_n^2 + \frac{k}{m}\right] = 0$$

Solve for ω_n

$$\omega_n = \sqrt{\frac{k}{m}} = 1.5\ \frac{rad}{s}$$

Solve for A

$$x(0) = Asin\left(\sqrt{\frac{k}{m}}t + \phi\right) = Asin(\phi) = 0.4 \ \rightarrow \ A = \frac{x_0}{\sin(\phi)} = \frac{0.4}{\sin(\phi)}$$

Solve for ϕ

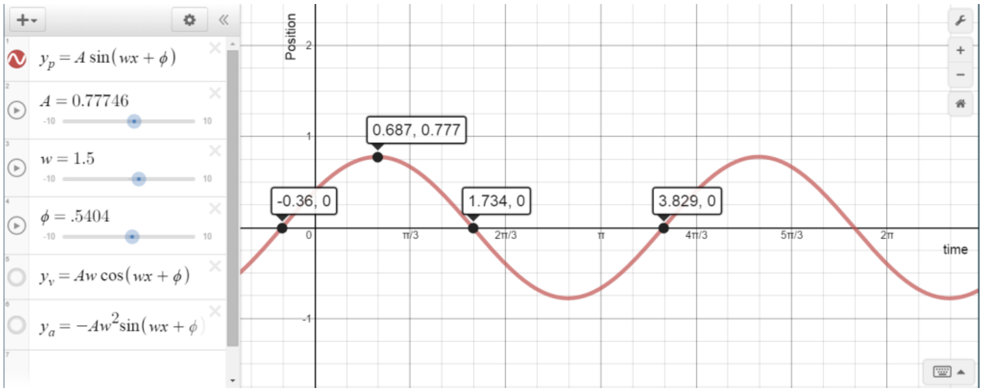
$$\dot{x}(0) = A\sqrt{\frac{k}{m}}cos\left(\sqrt{\frac{k}{m}}t + \phi\right) = A\sqrt{\frac{k}{m}}cos(\phi) = \frac{0.4}{\sin(\phi)}\sqrt{\frac{k}{m}}cos(\phi) = 0.4\sqrt{\frac{k}{m}}\cot(\phi) = 1$$

$$\rightarrow \cot(\phi) = \frac{1}{0.4\sqrt{\frac{k}{m}}} \quad \rightarrow \tan(\phi) = 0.4\sqrt{\frac{k}{m}} = 0.4\sqrt{\frac{225}{100}} = 0.6$$

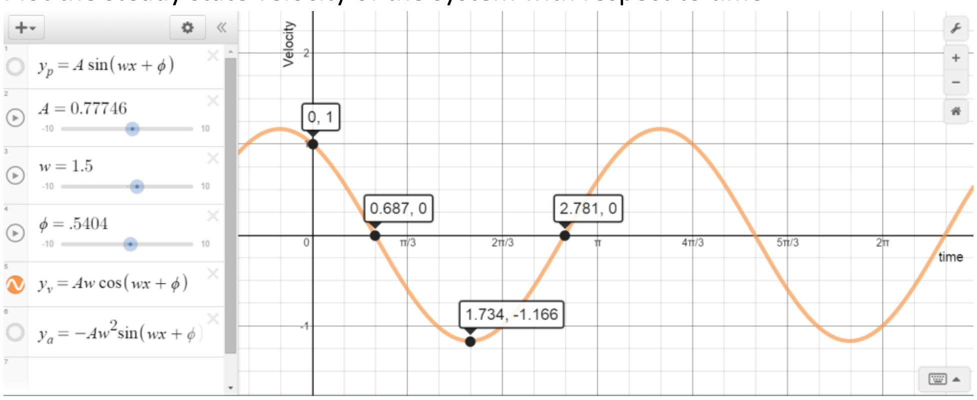
$$\rightarrow \phi = \tan^{-1}(0.6) = 0.5404$$
$$\rightarrow A = \frac{0.4}{\sin(0.5404)} = 0.77746$$

Homework 2

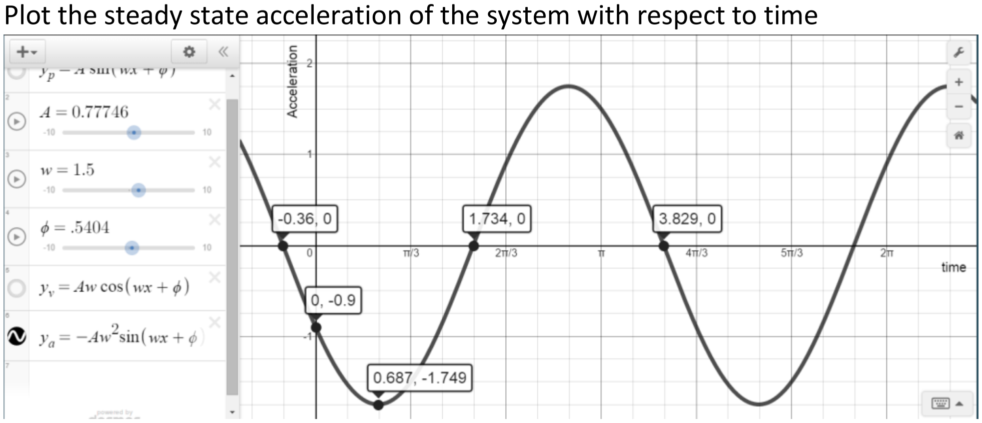
a) Plot the steady-state displacement of the system with respect to time.



b) Plot the steady state velocity of the system with respect to time



c) Plot the steady state acceleration of the system with respect to time



d) Compare and contrast the magnitude and phase of the displacement, velocity, and acceleration

a. The position and acceleration seem to be 90° out of phase with the velocity
b. The magnitude of acceleration > velocity > position
c. The acceleration of the mass returns to its initial value periodically, at the same point in time as the mass returns to its initial position

Problem 2

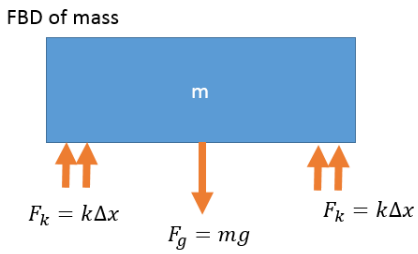
Consider that you are working as an engineer for the Boeing Corporation on the CH-47 Chinook. As observed in class, disturbances such as a shock or sudden impact can cause the aircraft to exhibit ground resonance. To avoid such an issue the landing gear is to be designed with a shock absorber spring system. Under its own mass the landing gear deflects 0.05 m. The landing gear is designed to be critically damped. The aircraft has four landing gear as depicted in the diagram.

a) Calculate the equivalent damping and stiffness constants of the system
b) Upon loading with troops, fuel, and armament the mass of the system increases by 50%. By calculations, determine how this increase in mass affects the damping ratio.

Because the landing gear is designed to be critically damped we can say that $\zeta = 1 = \frac{c}{2\sqrt{km}}$

From this equation we can see that $c = 2\sqrt{km}$

By creating a FBD of the mass of the Chinook we can equate forces



The sum of forces in y = $4k\Delta x - mg = 0$

From this relationship we can solve for the equivalent spring constant k_{eq} in terms of $m, g, \Delta x$. Because

$$k_{eq} = 4k$$

$$k_{eq} = \frac{mg}{\Delta x}$$

From the fact that the landing gear deflects 0.05 m under its own mass we can conclude

$$k_{eq} = 196.2\, m \text{ [N/m]}$$

From this we can find the damping ratio by

$$c = 2\sqrt{km} = 2 * \sqrt{196.2}\, m \cong 28m$$

If the mass is increased by 50% the damping ratio would change by

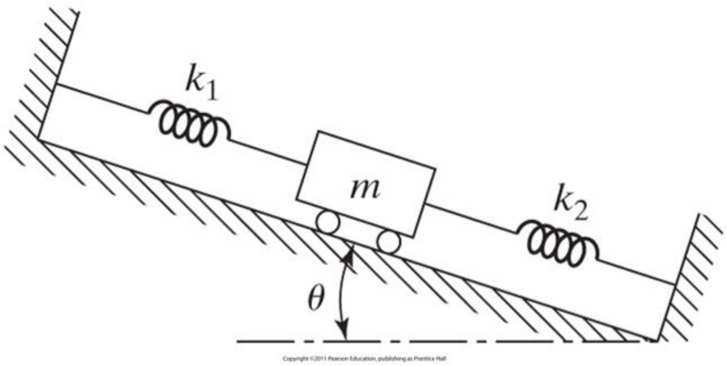
$$\zeta = \frac{c}{2\sqrt{km}} = \frac{28m_{old}}{2\sqrt{196.2m_{old} * m_{new}}} \quad \text{if} \quad m_{new} = 1.5m_{old} \quad \text{then}$$

$$\zeta = \frac{28m_{old}}{2\sqrt{196.2m_{old} * 1.5m_{old}}} = \frac{28m_{old}}{2m_{old}\sqrt{196.2 * 1.5}} \cong \frac{14}{17.15} \cong 0.816$$

This means that with the added weight the chinook would be underdamped

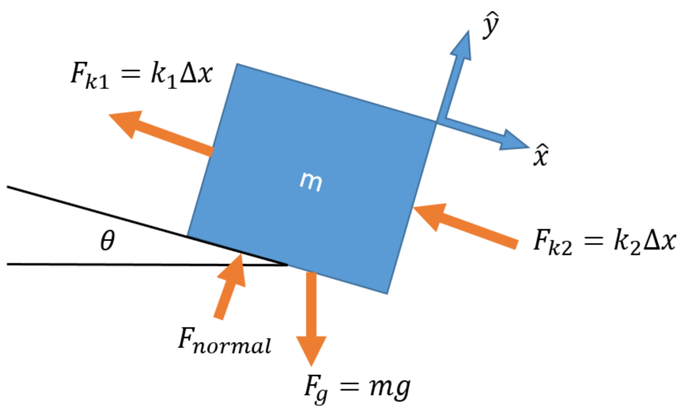
Problem 3

Determine the equation of motion and find the natural frequency of vibration of a spring--- mass System arranged on an inclined plane, as shown below. Explain what effect gravity has on the equation of motion and the system's natural frequency.



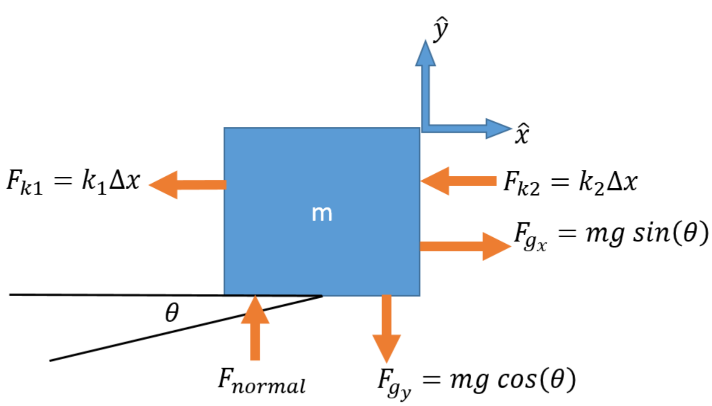
Assumption : no friction between the mass and the surface

Start with a FBD of the mass



To sum the forces we should first resolve F_g into the chosen x and y components

Redraw the FBD



From here we can see that $\sum F_x = mg\sin(\theta) - x(k_1 + k_2) = m\ddot{x}$

Putting the derived equation into standard form we get the equation of motion

$$\ddot{x} + \frac{(k_1 + k_2)}{m}x = g\sin(\theta)$$

To find the natural response we set the above equation to 0 which means there is no external force on the system

Using a trial solution of

$$x = Asin(\omega t + \phi)$$

$$\dot{x} = A\omega_n cos(\omega_n t + \phi)$$

$$\ddot{x} = -A\omega_n^2 sin(\omega_n t + \phi)$$

Substitute back in to get

$$A\sin(\omega_n t + \phi)\left[-\omega_n^2 + \frac{(k_1 + k_2)}{m}\right] = 0$$

Solve for $\omega_n = \sqrt{\frac{(k_1+k_2)}{m}}$

The force of gravity does not change the natural frequency of the vibration, but instead offsets the position at which the mass will vibrate. In this ideal example the mass will continue to vibrate indefinitely because there is no component in the system which takes energy away.

Problem 4

The amplitude of vibration of an undamped system is measured to be 1mm. the phase shift from t=0 is measured to be 2 rad and the frequency is found to be 5 rad/s. Calculate the initial conditions that caused this vibration to occur. Assume the response is of the form $x(t) = Asin(\omega_n t + \phi)$

$$A = 1\ [mm]$$

$$\phi = 2\ [rad]$$

$$\omega_n = 5\ \left[\frac{rad}{s}\right]$$

$$x(t) = A\sin(\omega_n t + \phi) = 1 \cdot \sin(5t + 2) = 0.909$$

$$\dot{x}(t) = A\omega_n\cos(\omega_n t + \phi) = 1 \cdot 5 \cdot \cos(5t + 2) = -2.08$$

Problem 5

Using complex analysis, derive the solution of $m\frac{d^2x}{dt^2}+k\frac{dx}{dt}=0$ and plot the result using MATLAB for at least two periods for the case that the angular natural frequency is $2\frac{rad}{s}$, initial position is 1mm, and initial velocity is $\sqrt{5}\frac{mm}{s}$.

From the given EOM we can use a general trial solution of

$$x = ae^{\lambda t}$$

$$\dot{x} = a\lambda e^{\lambda t}$$

$$\ddot{x} = a\lambda^2 e^{\lambda t}$$

Substitution

$$ae^{\lambda t}[m\lambda^2 + k\lambda] = 0$$

The Characteristic equation is therefore

$$\lambda^2 + \frac{k}{m}\lambda = 0$$

We can find the roots of this equation with the quadratic equation

$$\lambda_{1,2} = \pm\sqrt{-\frac{k}{m}} = \pm j\sqrt{\frac{k}{m}} \quad where\ j = \sqrt{-1}$$

Plug this result into the trial solution to get

$$x = a_1 e^{j\sqrt{\frac{k}{m}}} + a_2 e^{-j\sqrt{\frac{k}{m}}}$$

Using eulers identity $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$

$$x = a_1\left(cos\left(\sqrt{\frac{k}{m}}\,t\right) + j\,sin\left(\sqrt{\frac{k}{m}}\,t\right)\right) + a_2\left(cos\left(\sqrt{\frac{k}{m}}\,t\right) - j\,sin\left(\sqrt{\frac{k}{m}}\,t\right)\right)$$

Simplify and combine

$$x = \cos\left(\sqrt{\frac{k}{m}}\,t\right)(a_1 + a_2) + \sin\left(\sqrt{\frac{k}{m}}\,t\right)(j(a_1 - a_2))$$

Homework 2

Let

$$A_1 = a_1 + a_2$$

$$A_2 = j(a_1 + a_2)$$

$$x = A_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + A_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

Solve for A_1 & A_2 using the boundary conditions

$$x(0) = 1\ [mm] = A_1$$

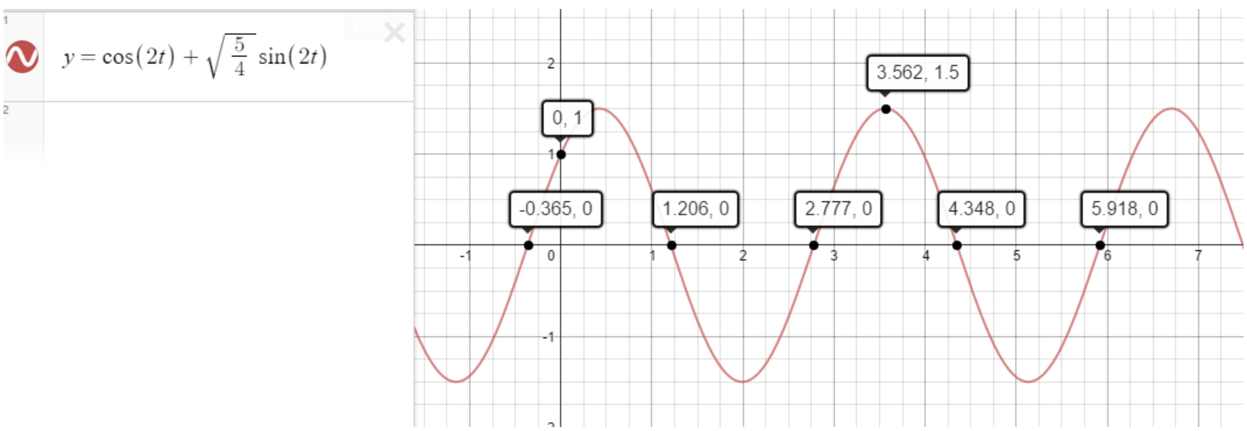
$$\dot{x}(0) = \sqrt{5}\ \left[\frac{mm}{s}\right] = A_2 \sqrt{\frac{k}{m}}$$

Because $\sqrt{\frac{k}{m}}$ is equal to the natural frequency

$$A_2 = \frac{\sqrt{5}}{2}\ \left[\frac{s}{rad} \cdot \frac{mm}{s}\right]$$

Therefore the position as a function of time is

$$x = \cos(2t) + \frac{\sqrt{5}}{2}\sin(2t)$$



Problem 6

Consider a spring-mass-damper system with equation of motion given by $\ddot{x} + 2\dot{x} + 2x = 0$. Calculate the damping ratio and determine if the system is overdamped, underdamped, or critically damped.

EOM

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$m = 1$$

$$c = 2$$

$$k = 2$$

Trial solution

$$x = ae^{\lambda t}$$

$$\dot{x} = a\lambda e^{\lambda t}$$

$$\ddot{x} = a\lambda^2 e^{\lambda t}$$

Substitute

$$ae^{\lambda t}[m\lambda^2 + c\lambda + k] = 0$$

Characteristic equation (because$ae^{\lambda t} \neq 0$)

$$m\lambda^2 + c\lambda + k = 0$$

Find the roots

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{4mk}{4m^2}} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

Define the damping ratio

$$\zeta = \frac{damping\ coeficient}{critically\ damped\ case} = \frac{c}{c_{cr}}$$

c_{cr} is the case there the term under the radical for the eigenvalues =0

$$\frac{c^2}{4m^2} - \frac{k}{m} = 0 \quad \rightarrow \quad c^2 = 4km \quad \rightarrow c_{cr} = 2\sqrt{km}$$

Substitute in known values

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2} = 0.707$$

Because $0 < \zeta < 1$ the s-m-d system is under damped.