Problem 1
A viscously damped system has a stiffness of 5,000 N/m, critical damping constant of 0.2 N-s/mm, and a logarithmic decrement of 2.0. If the system is given an initial velocity of 1 m/s, determine the maximum displacement of the system.

$$k = 5000\frac{N}{m}$$

$$c_{cr} = 0.2\frac{Ns}{mm} \cdot 1000\frac{mm}{m} = 200\frac{Ns}{m}$$

$$\delta = 2.0$$

$$v_0 = 1\frac{m}{s}$$

Find : max displacement

Solve for ζ

$$\zeta = \frac{2}{\sqrt{4\pi^2 + 4}} = 0.3033$$

Solve for c

$$0.3033 = \frac{c}{c_{cr}} = \frac{c}{200} \quad \rightarrow c = 60.66$$

Solve for m

$$c_{cr} = 2\sqrt{km} \quad \rightarrow \quad c_{cr}^2 = 4km \quad \rightarrow \quad m = \frac{c_{cr}^2}{4k} = 2\left[\frac{N^2 s^2}{m^2}\left|\frac{m}{N}\right|\right] = 2\,[kg]$$

Solve for ω_n

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5000}{2}} = 50$$

Solve for ω_d

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 50 \cdot 0.95289 = 47.6455$$

Assume the system starts with no initial displacement

$$x_0 = 0$$

$$x(t) = e^{-\zeta\omega_n t}[(A)\sin(\omega_d t + \phi)]$$

$$A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (\omega_d x_0)^2}{\omega_d^2}} = \sqrt{\frac{(1 + \zeta\omega_n x_0)^2 + (\omega_d x_0)^2}{\omega_d^2}} = 0.02099$$

$$\phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right)$$

But because the maximums do not occur when sine = 1, due to the exponential term, we need to find where the function is a maximum by setting the velocity to 0 and solving for time

$$\dot{x}(t) = -\zeta\omega_n e^{-\zeta\omega_n t} Asin(\omega_d t) + \omega_d e^{-\zeta\omega_n t} Acos(\omega_d t) = 0$$

$$\omega_d e^{-\zeta\omega_n t} Acos(\omega_d t) = \zeta\omega_n e^{-\zeta\omega_n t} Asin(\omega_d t)$$

$$\frac{\omega_d}{\zeta\omega_n} = \frac{sin(\omega_d t)}{cos(\omega_d t)} = \tan(\omega_d t)$$

$$\omega_d t = \tan^{-1}\left(\frac{\omega_d}{\zeta\omega_n}\right)$$

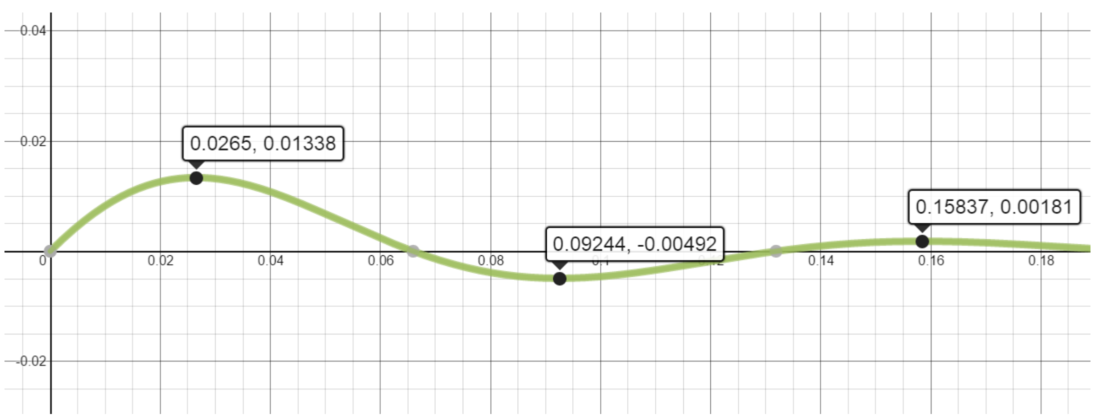
$$t = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{\zeta\omega_n}\right) = \frac{1}{47.65}1.2626 = 0.0265$$

There for the max amplitude, when initial displacement is zero is

$$x(t) = e^{-\zeta\omega_n t} Asin(\omega_d t)$$

$$x(0.0265) = 0.0134$$

A plot of this function reveals the true max amplitude



Problem 2

A body vibrating with viscous damping makes five complete oscillations per second, and in 50 cycles its amplitude diminishes to 10 percent. Determine the logarithmic decrement and the damping ratio.

Convert the given information about frequency

$$5\frac{oscillations}{second} \rightarrow \frac{1}{5}(s) = T \rightarrow \frac{2\pi}{\omega_d} = 2\pi T^{-1} \rightarrow \omega_d = 10\pi$$

From the information given about diminishing to 10% in 50 cycles we can write

$\ln\left(\frac{x_0}{x_1}\right) = \ln\left(\frac{A_0}{0.1A_0}\right)$ this holds because its exactly 50 cycles so the amplitude must be on the same point of the sinusoid

$$\zeta = \frac{\ln(10)}{\sqrt{(2\pi)^2 + \ln(10)^2}} = 0.344$$

Problem 3

Do problem 1.65 from the text.

1.65. Calculate the frequency of the compound pendulum of Figure P1.65 if a mass m_T is added to the tip, by using the energy method. Assume the mass of the pendulum is evenly distributed so that its center of gravity is in the middle of the pendulum of length l.

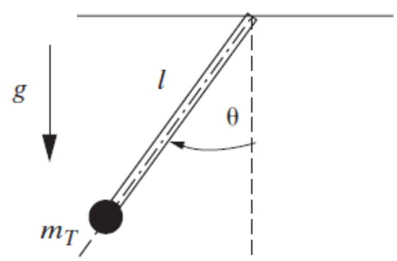
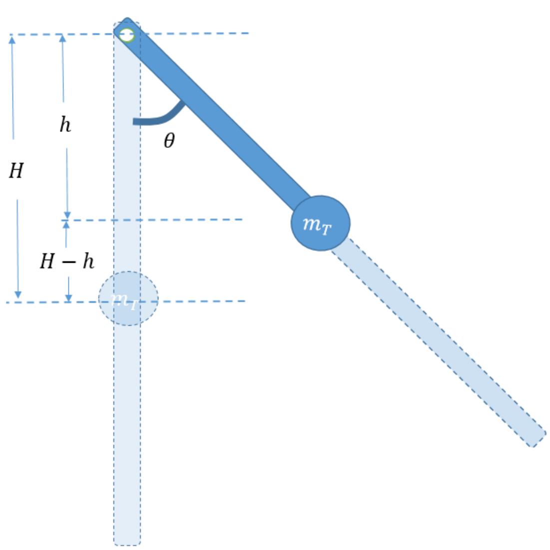


Figure P1.65 A compound pendulum with a tip mass.



From this simplified model we can find the energy in the system

We first want to find the potential energy

$$P_e = mgz$$

The potential energy at any given time is a function of theta, which comes about through the height $z = (H - h)$

$$h = Hcos(\theta)$$

H is known to be the length from the pin to the center of mass

$$H = \frac{L}{2}$$

The potential energy as a function of theta is therefore

$$P_e(\theta) = mg(H - h) = mgH - mgHcos(\theta) = \frac{mgL}{2} - \frac{mgL}{2}\cos(\theta)$$

Now we should find the kinetic energy of the mass

$$K_e = \frac{1}{2}mV^2$$

Because the mass is rotating about a fixed point the velocity V is actually angular velocity

$$V = \dot{\theta} \quad \& \quad m \rightarrow J_0$$

So the kinetic energy becomes

$$K_e = \frac{1}{2}J_0\dot{\theta}^2$$

Due to conservation of energy

$$P_e + K_e = constant$$

$$\frac{mgL}{2} - \frac{mgL}{2}\cos(\theta) + \frac{1}{2}J_0\dot{\theta}^2 = constant$$

We can take the derivative of the energy equation to say that the change in energy of each component must equal each other

$$\frac{d}{dt}\left(\frac{mgL}{2} - \frac{mgL}{2}\cos(\theta) + \frac{1}{2}J_0\dot{\theta}^2\right) = \frac{d}{dt}(constant)$$

$$\frac{mgL}{2}\dot{\theta}\sin(\theta) + \frac{1}{2}J_0 2\dot{\theta}\ddot{\theta} = 0$$

Divide out the $\dot{\theta}$ term

$$\frac{mgL}{2}\sin(\theta) + J_0\ddot{\theta} = 0$$

We can see that this equation is non-Linear so we must linearize it by constraining the θ term to be very small

$$\theta \approx \sin(\theta)$$

$$\frac{mgL}{2}\theta + J_0\ddot{\theta} = 0$$

From this EOM we can find the standard form

$$\ddot{\theta} + \frac{mgL}{2J_0}\theta = 0$$

Substitute the moment J_0 for a rotating rod with one end fixed

$$J_0 = \frac{mL^2}{3}$$

$$\ddot{\theta} + \frac{mgL}{2\frac{mL^2}{3}}\theta = 0$$

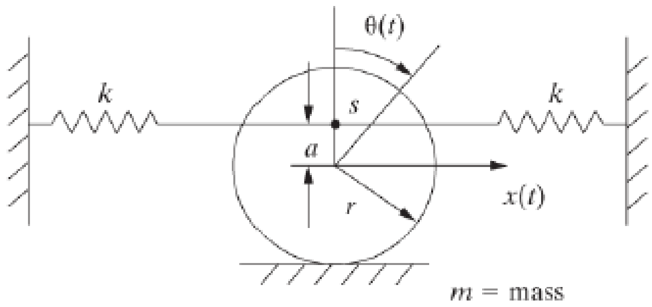
$$\ddot{\theta} + \frac{3g}{2L}\theta = 0$$

Therefore the natural frequency is

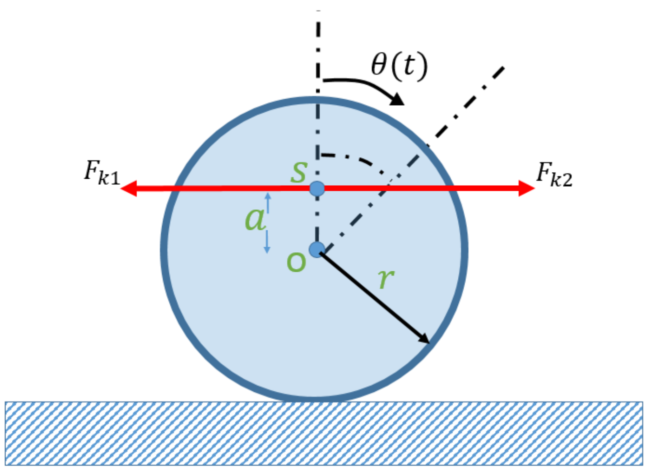
$$\omega_n^2 = \frac{3g}{2L} \quad \rightarrow$$

$$\omega_n = \sqrt{\frac{3g}{2L}}$$

Problem 4
Use the energy method to derive the equation of motion in terms of $\theta(t)$ and the natural frequency for the system shown below.



Begin with a FBD



From this picture we can see that there is potential energy being added off axis.

$$U_s = 2\left(\frac{1}{2}kx_s^2\right)$$

The rotation of the disk has rotational energy

$$T_{rot} = \frac{1}{2}J\dot{\theta}^2$$

The mass has translational energy

$$T_{trans} = \frac{1}{2}m\dot{x_0}^2$$

The sum of the energies must be constant because there are no known losses in this system.

$$kx_s^2 + \frac{1}{2}m\dot{x_O}^2 + \frac{1}{2}J\dot{\theta}^2 = const$$

Because there is a no slip condition we can relate the arc length to the distance that point O travels

$$r\theta = x_O \quad \rightarrow \quad \theta = \frac{x_O}{r}$$

Notice though that point s travels a distance more than point O because it is off axis, this extra distance is

$$x_a = \theta a$$

So the displacement on the spring is therefore

$$x_s = x_a + x_O = \theta a + \theta r = \theta(a + r)$$

Replace θ in the energy balance equation

$$k(a+r)^2\theta^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}J\dot{\theta}^2 = const$$

Take the derivative of the energy balance WRT θ

$$2k(a+r)^2\theta\dot{\theta} + \frac{1}{2}m2r^2\dot{\theta}\ddot{\theta} + \frac{1}{2}J2\dot{\theta}\ddot{\theta} = 0$$

Factor out $\dot{\theta}$ and simplify

$$2k(a+r)^2\theta + mr^2\ddot{\theta} + J\ddot{\theta} = 0$$

Substitute the rotational momentum

$$J = \frac{1}{2}mr^2$$

Simplify

$$2k(a+r)^2\theta + \frac{3}{2}mr^2\ddot{\theta} = 0$$

Standard form

$$\frac{2k(a+r)^2}{\frac{3}{2}mr^2}\theta + \ddot{\theta} = 0$$

Simplify

$$EOM \quad \ddot{\theta} + \omega_n^2\,\theta = 0$$

$$\omega_n = \sqrt{\frac{4k}{3m}\frac{(a+r)^2}{r^2}}$$

Problem 5

Research or read in a text about the topic of vibrational stability. Define in 3-4 sentences for each term and in your own words (do not copy from any source): flutter, divergent instability, and self-excited vibrations.

Flutter : an underdamped system that has energy added proportional to the dependent variable term or the derivative of the dependent variable term. (negative c or k)

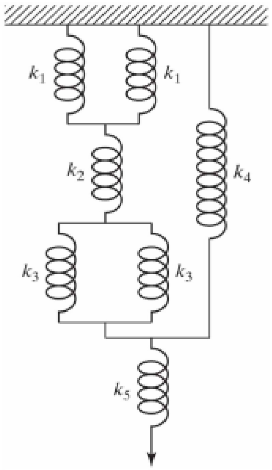
Divergent : an overdamped / critically damped system that has energy added proportional to the dependent variable term or the derivative of the dependent variable term. (negative c or k)

Instability : the system is infinitely transient, and tends to $\pm$ infinity as $\lim_{t\to\infty}$

Self-excited vibrations : energy is added to the system by internal system components

Problem 6

Determine the single equivalent spring constant for the system shown below.

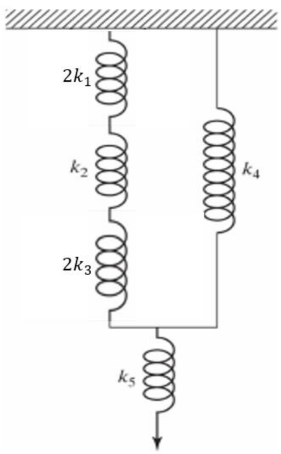


Because k_1 & k_1 , k_3 & k_3 are in parallel WRT each other they can be combined into

$$k_1 \;\&\; k_1 = 2k_1$$

$$k_3 \;\&\; k_3 = 2k_3$$

This result can be seen on the equivalent spring system to below



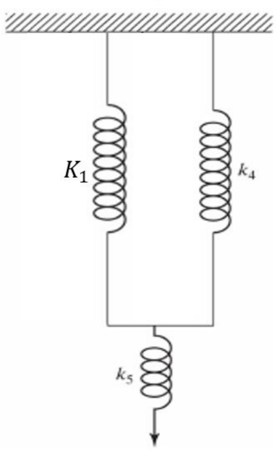
Because $2k_1$ & k_2 & $2k_3$ are in series they can be rewritten into an equivalent spring called K_1

First two springs together

$$k_{eq_1} = \frac{2k_1k_2}{2k_1 + k_2}$$

Now add the next in the series

$$\frac{k_{eq_1}2k_3}{k_{eq_1} + 2k_3}$$

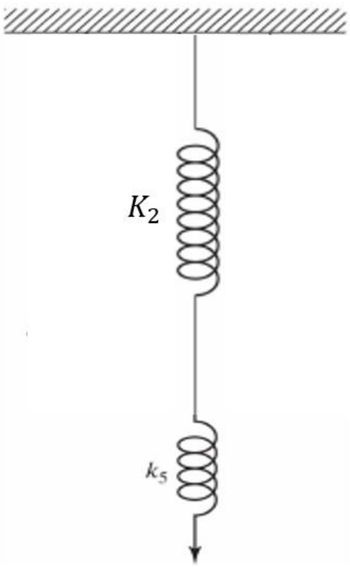


Sub in

$$\frac{\frac{2k_1k_2 2k_3}{2k_1 + k_2}}{\frac{2k_1k_2}{2k_1 + k_2} + 2k_3}$$

Simplify the numerator and denominator

$$\frac{\frac{4k_1k_2k_3}{2k_1 + k_2}}{\frac{2k_1k_2 + 2k_3(2k_1 + k_2)}{2k_1 + k_2}} \rightarrow \frac{4k_1k_2k_3}{\cancel{2k_1 + k_2}} \frac{\cancel{2k_1 + k_2}}{2k_1k_2 + 2k_3(2k_1 + k_2)}$$



Expand and simplify

$$K_1 = \frac{4k_1k_2k_3}{2k_1k_2 + 4k_3k_1 + 2k_3k_2}$$

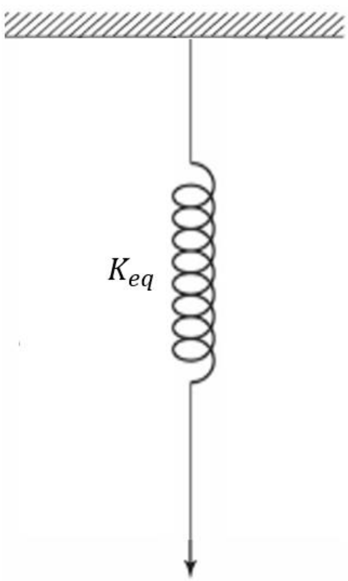
Notice that K_1 & k_4 are in parallel so they can be added

$$K_2 = \frac{4k_1k_2k_3}{2k_1k_2 + 4k_3k_1 + 2k_3k_2} + k_4$$

Finally K_2 & k_5 are in series

$$\frac{\left(\frac{4k_1k_2k_3}{2k_1k_2 + 4k_3k_1 + 2k_3k_2} + k_4\right)k_5}{\left(\frac{4k_1k_2k_3}{2k_1k_2 + 4k_3k_1 + 2k_3k_2} + k_4\right) + k_5}$$

Expand



$$\frac{\frac{4k_1k_2k_3k_5 + k_5k_4(2k_1k_2 + 4k_3k_1 + 2k_3k_2)}{2k_1k_2 + 4k_3k_1 + 2k_3k_2}}{\frac{4k_1k_2k_3 + k_4(2k_1k_2 + 4k_3k_1 + 2k_3k_2)}{2k_1k_2 + 4k_3k_1 + 2k_3k_2} + k_5}$$

$$\frac{\frac{4k_1k_2k_3k_5 + k_5k_4(2k_1k_2 + 4k_3k_1 + 2k_3k_2)}{2k_1k_2 + 4k_3k_1 + 2k_3k_2}}{\frac{4k_1k_2k_3 + k_4(2k_1k_2 + 4k_3k_1 + 2k_3k_2) + k_5(2k_1k_2 + 4k_3k_1 + 2k_3k_2)}{2k_1k_2 + 4k_3k_1 + 2k_3k_2}}$$

$$\frac{4k_1k_2k_3k_5 + k_5k_4(2k_1k_2 + 4k_3k_1 + 2k_3k_2)}{4k_1k_2k_3 + k_4(2k_1k_2 + 4k_3k_1 + 2k_3k_2) + k_5(2k_1k_2 + 4k_3k_1 + 2k_3k_2)}$$

Expand Again

$$\frac{4k_1k_2k_3k_5 + 2k_1k_2k_5k_4 + 4k_3k_1k_5k_4 + 2k_3k_2k_5k_4}{4k_1k_2k_3 + 2k_1k_2k_4 + 4k_3k_1k_4 + 2k_3k_2k_4 + 2k_1k_2k_5 + 4k_3k_1k_5 + 2k_3k_2k_5}$$

Rearrange spring coefficients to be in order so we may see if combinations are available

$$K_{eq} = \frac{4k_1k_2k_3k_5 + 2k_1k_2k_4k_5 + 4k_1k_3k_4k_5 + 2k_2k_3k_4k_5}{4k_1k_2k_3 + 2k_1k_2k_4 + 2k_1k_2k_5 + 4k_1k_3k_4 + 2k_2k_3k_4 + 4k_1k_3k_5 + 2k_2k_3k_5}$$