Vibrations Alexander McGhee

Problem 1

For a base motion system described by

$$m\ddot{x} + c\dot{x} + kx = cY\omega_b \cos\omega_b t + kY\sin\omega_b t$$

with $m = 100$ kg, $c = 50$ kg/s, $k = 1000$ N/m, $Y = 0.03$ m, and $\omega_b = 3$ rad/s, compute the magnitude of the particular solution. Last, compute the transmissibility ratio.

The particular solution assumes the following form

$$x_{p1}(t) = \frac{2\zeta\omega_n\omega_b Y}{\sqrt{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}}\cos(\omega_b t - \theta_1)$$

$$x_{p2}(t) = \frac{\omega_n^2 Y}{\sqrt{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}}\sin(\omega_b t - \theta_1)$$

By POS the two trial solutions can be combined, and simplified by trig identities into the following form

$$x_p(t) = \omega_n Y\sqrt{\frac{\omega_n^2 + (2\zeta\omega_b)^2}{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}}\cos(\omega_b t - \theta_1 - \theta_2)$$

The magnitude of the particular solution can be seen as the collection of terms in front of the cosine term.

$$\omega_n Y\sqrt{\frac{\omega_n^2 + (2\zeta\omega_b)^2}{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}}$$

We need to find the following terms

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{100}} = 3.1623$$

$$\omega_b = given = 3$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{50}{2 \cdot 100 \cdot 3.1623} = 0.0791$$

Plug these values into the magnitude equation

$$3.1623 \cdot 0.03\sqrt{\frac{3.1623^2 + (2 \cdot 0.0791 \cdot 3)^2}{(3.1623^2 - 3^2)^2 + (2 \cdot 0.0791 \cdot 3.1623 \cdot 3)^2}} = 0.0949\sqrt{\frac{10.22}{1 + 2.2525}} = 0.1682$$

The transmissibility ratio is the magnitude divided by the max forcing displacement Y

$$\frac{X}{Y} = \frac{0.1682}{3} = 0.0561$$

Problem 2

A vibrating mass of 300 kg mounted on a massless support by a spring of stiffness 40,000 N/m and a damper of unknown damping coefficient is observed to vibrate with a 10-mm amplitude while the support vibration has a maximum amplitude of only 2.5 mm (at resonance). Calculate the damping constant and the amplitude of the force on the base.

In this case we are given the tools needed to calculate the transmissibility (at resonance) which is

$$\frac{X}{Y} = \frac{10}{2.5}\left[\frac{mm}{mm}\right] = 4$$

Because the system is at resonance $\omega_n \approx \omega_b$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{40000}{300}} \cong 11.55$$

Furthermore the frequency ratio

$$r \cong 1$$

Using the transmissibility equation we can solve for ζ

$$\frac{X}{Y} = \sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}} \;\rightarrow\; \frac{X}{Y} = \sqrt{\frac{1+4\zeta^2}{4\zeta^2}} \;\rightarrow\; \left(\frac{X}{Y}\right)^2 = \frac{1+4\zeta^2}{4\zeta^2} \;\rightarrow$$

$$\zeta = \sqrt{\frac{1}{4\left(\left(\frac{X}{Y}\right)^2 - 1\right)}} = \sqrt{\frac{1}{4(16-1)}} = 0.1291$$

From ζ we can find the damping constant c

$$\zeta = \frac{c}{2m\omega_n} \quad\rightarrow\quad c = \zeta 2m\omega_n = 0.1291 \cdot 2 \cdot 300\,[kg] \cdot 11.55\left[\frac{rad}{s}\right] = 894.6\frac{kg}{s}$$

The displacement of the mass can be found using the following equation

$$x_p(t) = \omega_n Y\sqrt{\frac{\omega_n^2+(2\zeta\omega_b)^2}{(\omega_n^2-\omega_b^2)^2+(2\zeta\omega_n\omega_b)^2}}\cos(\omega_b t - \theta_1 - \theta_2)$$

We need the second derivative which is the particular acceleration. Using $F = ma$ we can find the force

$$F = m\ddot{x}_p(t) = m\omega_b^2\omega_n Y\sqrt{\frac{\omega_n^2+(2\zeta\omega_b)^2}{(\omega_n^2-\omega_b^2)^2+(2\zeta\omega_n\omega_b)^2}}\cos(\omega_b t - \theta_1 - \theta_2) = kYr^2\frac{X}{Y}\cos(\rightarrow 0)$$

The amplitude of the force can be found when the cosine term $\rightarrow \pm 1$

$$F = kY\frac{X}{Y} = kX = 40000\left[\frac{N}{m}\right]\cdot 10\,[mm]\cdot 10^{-3}\left[\frac{m}{mm}\right] = 400\,[N]$$

Problem 3

Compare and contrast in at least a half page essay resonance for undamped and damped spring–mass system. Be sure to compare with the regards to amplitude and phase of the vibration.

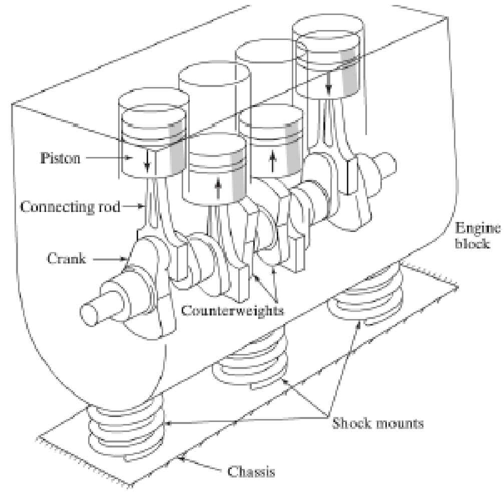
The main difference between the spring mass and spring-mass-damper systems is the ability for the system to lose energy, which can only happen in the case of damping.

For the spring-mass case the forcing term will add energy to the system without a corresponding loss. This means that at resonance the forcing function will cause the system to have unbounded excitation. This is because the forced response is in phase with the natural response. When the natural frequency does not match the forcing frequency, there is a beat phenomenon where there is a periodic increase in vibrational amplitude followed by a periodic decrease. The amplitude of this beat is determined by the ratio of the natural frequency and the forced frequency.

In the case were we introduce damping, even for cases where the damping is very small, the unbounded excitation goes away, and instead we find that there is a transient period where the system will eventually settle to a steady state response that is some combination of the forced and natural response. The time it takes for the system to settle is determined by the homogenous response, the frequency ratio and the damping ratio. Therefore, for the case of damped systems the amplitude will eventually go to a steady state which can be viewed as the ratio of the energy loss to energy gain of the system.

Problem 4

A four-cylinder automobile engine is to be supported on three shock mounts, also known as *motor mounts*, as indicated in the figure to the right. The engine-block assembly weighs 500 lbs. If the unbalanced force generated by the engine is measured to be $200\sin 100\pi t$ lbs., design the three shock mounts, (each of stiffness k and viscous-damping constant c) such that the amplitude of vibration is less than 0.1 in. In other words, determine k and c subject to the design specification of amplitude of vibration.



$m = 500\ lb_f = \frac{500}{32.2\cdot 12} = 1.294\ slug_{in}$
excitation $= 200\sin(100\pi t)\ [lb_f]$
Max vibrational amplitude = $x_{max} = 0.1\ [in]$
because the 3 springs dampers are in parallel the equivalent stiffness is
$k_{eq} = 3k$ equivalent damping $c_{eq} = 3c$

EOM

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{200}{1.294}\sin(100\pi t)$$

The homogeneous solution is as follows

$$x_h(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

Where $\zeta \geq 1$ which would be the best case for a low amplitude response

$$\omega_d = \omega_n\sqrt{\zeta^2 - 1}$$

The particular solution is

$$x_p(t) = Xsin(\omega t - \theta)\ \left\{X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}, \theta = \tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)\right\}$$

Because the homogeneous portion will be a transient response we will focus on finding a max amplitude based on the particular response only such that the X term must be less than 0.1 $[in]$ assume $\zeta = 1$

$$\frac{f_0}{\sqrt{(k-\omega^2)^2+(c\omega)^2}} = \frac{154.5}{\sqrt{\left(\frac{k}{m}-314.15^2\right)^2+\left(\frac{c}{m}\cdot 314.15\right)^2}} = 0.1$$

$$\frac{k^2}{m^2} - 2\cdot\frac{314.15^2}{m}\,k + 314.15^4 + c^2\cdot\frac{314.15^2}{m^2} = \left(\frac{154.5}{0.1}\right)^2$$

$$k^2 - 2\cdot 1.294\cdot 314.15^2\,k + 314.15^4 + c^2\cdot 314.15^2 = 1.294^2\left(\frac{154.5}{0.1}\right)^2$$

$$k^2 - 255410\,k + c^2\cdot 98690 + 9735763097 = 0$$

$$k = 127705 \pm \frac{1}{2}\sqrt{6.5_{E10} - 394760c^2 - 3.89_{E10}} \;\rightarrow$$

$$k = \; 127705 \pm \sqrt{98690c^2 + 6.57_{E9}}$$

Problem 5

A spring-mass system, with mass of 100 kg and stiffness of 400 N/m, is subjected to a harmonic force $F(t) = 10\cos\omega t$ N. Find the steady-state response of the system when ω is equal to (a) 2 rad/s, (b) 0.2 rad/s, and (c) 20 rad/s. Discuss the results.

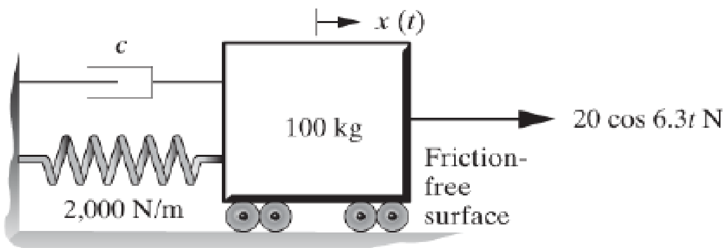
$$\omega_n = 2$$

a) There is no steady state response because the driving frequency matches the natural frequency, so the response goes to infinity
b) $x_{ss} = 0.0253\cos(0.2t)$
c) $x_{ss} = 0.000253\cos(20t)$

The difference in the responses from b, c are negligible because they are both an order of 10 magnitude away from the natural frequency and thus have a similar response. The outlier is part a which is equal to the natural frequency and thus explodes the response to infinity.

Problem 6

Compute a value of the damping coefficient c such that the steady-state response amplitude of the system in the figure below is 0.01 m.



$$\omega_n \cong 4.47$$

The particular solution is

$$x_p(t) = X cos(\omega t - \theta) \ \left\{ X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}, \theta = \tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right) \right\}$$

Because the homogeneous portion will be a transient response we will focus on finding a max amplitude based on the particular response only such that the X term must be less than 0.01 $[m]$

$$\frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + \left(\frac{c}{m}\omega\right)^2}} = \frac{0.2}{\sqrt{(4.47^2 - 6.3^2)^2 + \left(\frac{c}{m} \cdot 4.47\right)^2}} = 0.01$$

Solve for c

$$\left(\frac{0.2}{0.01}\right)^2 = 387.69 + \frac{20}{100^2}c^2 \ \rightarrow c = 100\sqrt{\frac{400 - 387.69}{20}} = 78.45\frac{kg}{s}$$