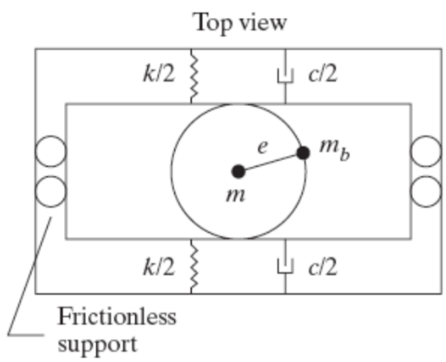
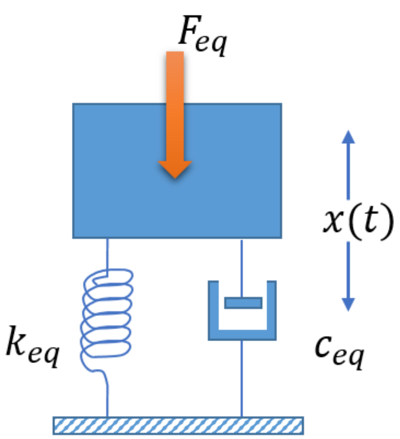
## Problem 1




A simplified model of a washing machine is illustrated in the side figure. A bundle of wet clothes forms a mass of 10 kg ($m_b$) in the machine and causes a rotating unbalance. The rotating mass is 20 kg (including $m_b$) and the diameter of the washer basket ($2e$) is 50 cm. Assume that the spin cycle rotates at 300 RPM. Let $k$ be 1000 N/m and $\zeta = 0.01$. Calculate the force transmitted to the sides of the washing machine. Discuss the assumptions made in your analysis in view of what you might know about washing machines.

***Equivalent system***




$$k_{eq} = 2 \cdot \frac{1}{2} k = k = 1000 \left[\frac{N}{m}\right]$$
$$m_b = mass\ of\ clothes = 10\ [kg]$$
$$m_{total} = 20\ [kg]$$
$$\zeta = 0.01$$
$$\omega_r = 300 \left[\frac{rotation}{min}\right] \cdot \frac{1}{60} \left[\frac{min}{s}\right] \cdot 2\pi \left[\frac{rad}{rotation}\right] = 31.4$$
$$\omega_n = \sqrt{\frac{k_{eq}}{m_{tot}}} = \sqrt{\frac{1000}{20} \left[\frac{1}{s^2}\right]} = 7.07 \left[\frac{1}{s}\right]$$
$$r = \frac{\omega_r}{\omega_n} = 4.44$$
$$e = 25_{E-2}\ [m]$$
$$F_{eq} = m_b e \omega_r^2 \sin(\omega_r t)$$

Using the equation for fixed base force transmissibility

$$F_t = F_0 \frac{X}{Y} = m_b e \omega_r^2 \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}$$

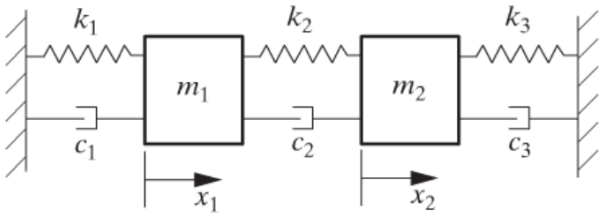
$$F_t = 10 \cdot 25_{E-2} \cdot 31.4^2 \sqrt{\frac{1 + (2 \cdot 0.01 \cdot 4.44)^2}{(1 - 4.44^2)^2 + (2 \cdot 0.01 \cdot 4.44)^2}}$$

$$F_t = 2465 \sqrt{\frac{1.00788}{350.2}} = 132.2\ [N]$$

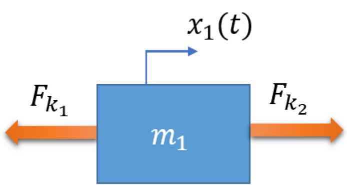
The major assumption that I'm making is that the clothes are evenly distributed in the machine, with the exception of the bundled mass of clothes that cause the imbalance. I am also making the assumption that the rotating potion of the washing machine is balanced and provided no mass contribution laterally.

**Problem 2**

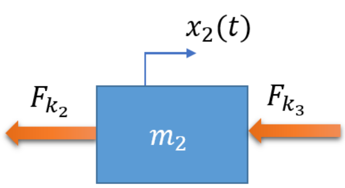
a) Show all steps to derive the equation of motion for the system below. Calculate the mass and stiffness matrices. Let $c_1 = c_2 = c_3 = 0$.




We can start by making a FBD for each mass then creating an equation of motion for each mass, each FBD will be created by assuming that the other mass remains fixed.




$$\sum F = -k_1x_1 + k_2(x_2 - x_1) = m_1\ddot{x}_1$$
$$m_1\ddot{x}_1 + x_1k_1 - k_2x_2 + k_2x_1 = 0$$
$$m_1\ddot{x}_1 + x_1(k_1 + k_2) - k_2x_2 = 0$$




$$\sum F = -k_2(x_2 - x_1) - k_3x_2 = m_2\ddot{x}_2$$
$$m_2\ddot{x}_2 + x_2k_3 + k_2x_2 - k_2x_1 = 0$$
$$m_2\ddot{x}_2 + x_2(k_3 + k_2) - k_2x_1 = 0$$

$$m_1\ddot{x}_1 + x_1(k_1 - k_2) + k_2x_2 = 0$$
$$m_2\ddot{x}_2 + x_2(k_3 + k_2) - k_2x_1 = 0$$

Set the previous equations into a matrix

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_3 + k_2) \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

b) Calculate the characteristic equation for the case:

$m_1 = 9$ kg, $m_2 = 1$ kg, $k_1 = 24$ N/m, $k_2 = 3$ N/m, $k_3 = 3$ N/m

First we need to use a trial solution for $\boldsymbol{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$

$$\boldsymbol{x} = ae^{j\omega t}\,\hat{\boldsymbol{u}} \quad \rightarrow \quad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = ae^{j\omega t}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\dot{\boldsymbol{x}} = aj\omega e^{j\omega t}\,\hat{\boldsymbol{u}} \quad \rightarrow \quad \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = aj\omega e^{j\omega t}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\ddot{\boldsymbol{x}} = -a\omega^2 e^{j\omega t}\,\hat{\boldsymbol{u}} \quad \rightarrow \quad \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = -a\omega^2 e^{j\omega t}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Next we can substitute into the matrix equations

$$-a\omega^2 e^{j\omega t}\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + a e^{j\omega t}\begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & (k_3+k_2) \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Divide throughout by $ae^{j\omega t}$

$$-\omega^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & (k_3+k_2) \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Because the two matrices have the same column vector , they can be combined

$$\begin{bmatrix} (k_1+k_2) - m_1\omega^2 & -k_2 \\ -k_2 & (k_3+k_2) - m_2\omega^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

If we call the square symmetric matrix $\boldsymbol{A}$ that multiplies $\hat{\boldsymbol{u}}$

$$\{\hat{\boldsymbol{u}}\} = \left[\boldsymbol{A}^{-1}\right]\{\boldsymbol{0}\}$$

Because the only result of this case is that $\hat{\boldsymbol{u}} = 0$ we can say that $\boldsymbol{A}$ is not invertible and must therefore be singular. It also follows that the $\det[\boldsymbol{A}] = 0$

$$\det\left[\begin{bmatrix} (k_1+k_2) - m_1\omega^2 & -k_2 \\ -k_2 & (k_3+k_2) - m_2\omega^2 \end{bmatrix}\right] = \left((k_1+k_2) - m_1\omega^2\right)\left((k_3+k_2) - m_2\omega^2\right) - k_2^2$$

$$(k_1+k_2)(k_3+k_2) - m_2\omega^2(k_1+k_2) - m_1\omega^2(k_3+k_2) + m_1m_2\omega^4 - k_2^2 = 0$$

Call

$$(k_1+k_2) = K_1$$
$$(k_3+k_2) = K_2$$

$$(K_1K_2 - k_2^2) - \omega^2(m_2K_1 + m_1K_2) + \omega^4(m_1m_2) = 0$$

Plug in knowns

$$\begin{Bmatrix} m_1 \\ m_2 \\ k_1 \\ k_2 \\ k_3 \end{Bmatrix} = \begin{Bmatrix} 9\ [kg] \\ 1\ [kg] \\ 24\ [N/m] \\ 3\ [N/m] \\ 3\ [N/m] \end{Bmatrix} \quad \rightarrow \quad \begin{Bmatrix} K_1 \\ K_2 \end{Bmatrix} = \begin{Bmatrix} k_1+k_2 \\ k_3+k_2 \end{Bmatrix} = \begin{Bmatrix} 27 \\ 6 \end{Bmatrix}[N/m]$$

$$(27\cdot 6 - 3^2) - \omega^2(1\cdot 27 + 9\cdot 6) + \omega^4(9\cdot 1) = 0$$

$$9\omega^4 - 81\omega^2 + 153 = 0$$

c) Solve for the system's natural frequencies.

Because the $\omega$ that we have been working with is the frequency due to the mass and spring we can say that it is the natural frequency $\omega_n$.

$$\omega_n^4 - 9\omega_n^2 + 17 = 0$$

$$\omega_n^2 = \frac{9}{2} \pm \frac{\sqrt{81 - 4 \cdot 17}}{2} = 4.5 \pm \frac{\sqrt{13}}{2}$$

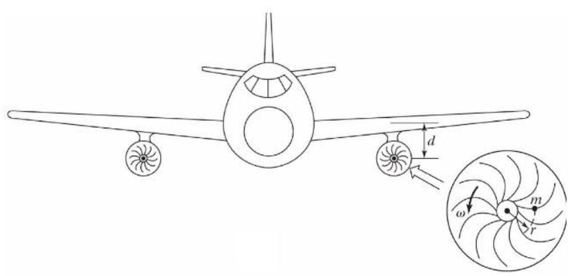
$$\omega_n = \sqrt{4.5 \pm \frac{\sqrt{13}}{2}}$$

$$\omega_{n_1} = 2.51$$
$$\omega_{n_2} = 1.64$$

**Problem 3**

An aircraft engine has a rotating unbalanced mass $m$ at radius $r$ (see figure below). The wing can be modeled as a cantilever beam of uniform cross section of width $a$ and height $b$. Assume the density of the wing is given by $\rho$. Assume damping and the effect of the wing between the engine and the free end to be negligible.

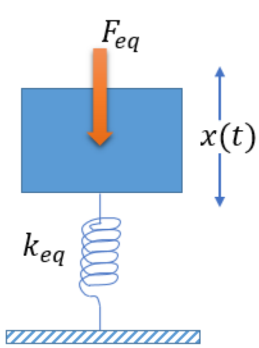
a) Derive an expression for the maximum deflection ($d$) of the engine at a speed of $N$ rpm.




Assumptions :

The deflection of the engine is in the $x$ direction only due to the small deflection relative to the distance from the aircrafts body.

Draw equivalent system.




$k_{eq} = \frac{3EI}{L^3}$ $\{this\ is\ the\ stiffness\ of\ the\ wing\ with\ L = body \rightarrow engine\}$

$F_{eq} = mr\omega^2 \sin(\omega t)$

Sum of forces = $m_{wing}\ddot{x} = -k_{eq}x + mr\omega^2 \sin(\omega t)$

EOM

$$\ddot{x} + \omega_n^2 x = mr\omega^2 \sin(\omega t)$$

Homework 8

Vibrations McGhee, Alexander

Solution

The max deflection occurs when the force due to the eccentric mass is pointing in the $\pm x$ direction. Assuming a steady state we can deduce the following.

$$x_p(max) = \frac{mr\omega^2}{m_{wing}(\omega_n^2 - \omega^2)}$$

b) Based on your answer to part a), find an aircraft of your choice in service today and solve for the maximum deflection of the wing.

Aircraft in service today King-air C90

$$m_{wing} = \rho_{aluminum} \cdot V_{aluminum} + m_{engine} + \frac{kg}{Gal_{fuel}} \cdot Gal_{fuel}$$

$2L = engine\ to\ engine = 12\ ft + 9\ in = 3.8862\ [m]$

$L = 1.943$

$E_{al} = 68.9\ GPa$

$t = 0.5048\ [m]$

$w = 1.852\ [m]$

Assuming the wing is full of gas and is considered solid

$$I = \frac{1}{12} L \cdot t^3 = 0.0208\ [m^4]$$

Engine (PT6A)

$m = 211.8\ kg$

$$\omega = 2000 \left[\frac{rev}{min}\right] \cdot \frac{1}{60} \left[\frac{min}{s}\right] \cdot 2\pi \left[\frac{rad}{rev}\right]$$

Use all of the above relations to solve the problem

$$k_{eq} = \frac{3EI}{L^3} = \frac{3 \cdot 68.9_{E9} \cdot 0.0208}{1.943^3} = 5.86_{E8} \left[\frac{N}{m}\right]$$

$$m_{tot} = (1.81 - 1.78)[m^3] \cdot 2720 \left[\frac{kg}{m^3}\right] + 211.8\ ]kg] + 0.81 \left[\frac{kg}{liter}\right] \cdot 0.264 \left[\frac{liter}{gal}\right] \cdot 106[gal]$$

$m_{tot} = 306[kg]$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{5.86_{E8}}{306}} = 1383 \left[\frac{rad}{s}\right]$$

$$\omega = 209 \left[\frac{rad}{s}\right]$$

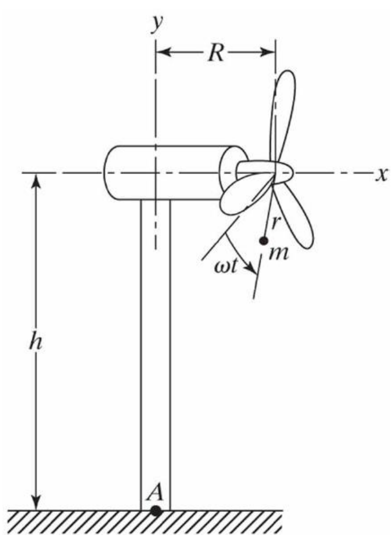
Assume the eccentric load is

$mr = 0.000125\ [kg \cdot m]$

$$\delta = \frac{mr\omega^2}{m_{wing}(\omega_n^2 - \omega^2)} = \frac{0.000125 \cdot 209^2}{306(1383^2 - 209^2)} = \frac{5.46}{5.7_{E8}} = 9.5_{E-9}\ [m]$$

**Problem 4**

"Green" energy sources will be relied upon to meet our future energy needs. Ocean-based energy sources, including under and above sea, are very promising. Google "undersea power generation" for interesting new developments/technologies. Consider an ocean-based three-bladed wind turbine as shown in the picture above, when rotating it has a small unbalanced mass *m* located at a radius *r* in the plane of the blades. The blades are located from the central vertical (*y*) axis at a distance *R*, and rotate at an angular velocity of $\omega$. The supporting column is a hollow steel shaft of outer diameter 0.1 m and inner diameter 0.08 m. Determine the maximum stresses developed at the base of the support (point A in the diagram below). Take into account imbalance in both *z* and *y* directions. The mass moment of inertia of the turbine system about the vertical (*y*) axis is 100 kg-m$^2$. Assume $R$=0.5 m, $m$=0.1 kg, $r$ = 0.1 m, $\omega$ = 31.416 rad/s, and $h$ = 8 m.




Small unbalanced mass

$m_0 = 0.1\ [kg]$ The mass of the unbalanced mass

$r = 0.1\ [m]$ Relative to the center of the blade

Blades

$R = 0.5\ [m]$ Distance from support shaft

$\omega_r = 31.416\ \left[\frac{rad}{s}\right]$ The angular velocity

Support column

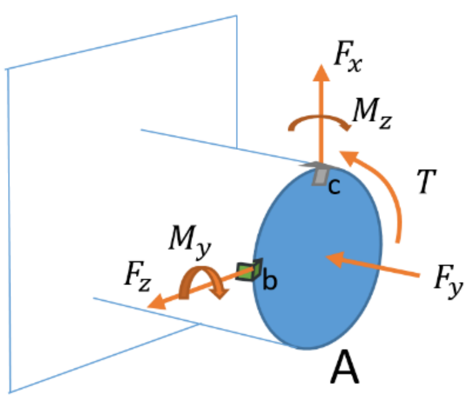
$E_{steel}$ The support is made of steel

$(r_o - r_i) = (0.1 - 0.08)\ [m]$ The support is hollow

$J = \frac{\pi}{2}(0.1^4 - 0.08^4) = 9.27_{E-5}$

$h = 8[m]$ The height of the support

$J_y = 100\ [kg \cdot m^2]$ The mass moment of inertia of the turbine system about the vertical y-axis




Find the max stress developed at the base of the support at point A in both $y$ & $z$ directions

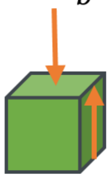
$$F_y = m_0 r \omega_r^2 \sin(\omega t)$$

$$F_z = m_0 r \omega_r^2 \cos(\omega t)$$

The stress caused by the off center mass will be torsional about the $y$-axis and torsional about the z axis

We can use relationships from mechanics of materials to solve

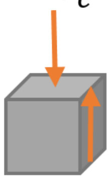
$$\sigma_b = \frac{F_y}{A} + \frac{M_z r}{I}$$



$$\tau_b = \frac{Tr}{J} + \frac{F_z A r}{It}$$

b

$$\sigma_c = \frac{F_y}{A} + \frac{M_x r}{I}$$



$$\tau_c = \frac{Tr}{J}$$

c

$$\sigma_b = \frac{m_0 r \omega_r^2}{\pi\left(r_o^2 - r_i^2\right)} + \frac{R m_0 r \omega_r^2 r_o}{I} = 875 + 742 = 1617$$

$$\tau_b = \frac{R m_0 r \omega_r^2 r_o}{J} + \frac{m_0 r \omega_r^2 A r_o}{I(r_o - r_i)} = 5338 + 838 = 6176$$

$$\sigma_{max} = \frac{\sigma_b}{2} + \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau_b^2} = 7.037\ [KPa]$$

$$\sigma_c = \frac{m_0 r \omega_r^2}{\pi\left(r_o^2 - r_i^2\right)} + \frac{h m_0 r \omega_r^2 r_o}{I} = 875 + 11872$$

$$\tau_c = \frac{R m_0 r \omega_r^2 r_o}{J} = 5338$$

$$\sigma_{max} = \frac{\sigma_b}{2} + \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau_b^2} = 112.9\ [KPa]$$

**Problem 5**

An air compressor, weighing 1000 lbs., and operating at 1500 rpm, is to be mounted on a suitable isolator to prevent unwanted vibration to the floor. A helical spring with a stiffness of 45,000 lb/in., another helical spring with a stiffness of 15,000 lb/in., and a shock absorber with a damping ratio of 0.15 are available for use. Design the isolation system to provide the best possible isolation using a combination of the springs and dampers provided.

$$m_{tot} = 1000\ [lb_m]$$
$$\omega_r = 1500 \left[\frac{rotation}{min}\right] \cdot \frac{1}{60}\left[\frac{min}{s}\right] \cdot 2\pi \left[\frac{rad}{rotation}\right] = 50\pi \left[\frac{rad}{s}\right]$$
$$k_1 = 45{,}000 \left[\frac{lb_f}{in}\right]$$
$$k_2 = 15{,}000 \left[\frac{lb_f}{in}\right]$$
$$\zeta_s = 0.15$$

Design an isolation system to provide the best isolation using the combination of springs and damper

4 combinations of springs are possible

$k_1$ only $\rightarrow \omega_n = \sqrt{\frac{k_1}{m}} = \sqrt{\frac{45{,}000}{32.2 \cdot 12 \cdot 1000}} = 0.341$

$k_2$ only $\rightarrow \omega_n = \sqrt{\frac{k_1}{m}} = \sqrt{\frac{15{,}000}{32.2 \cdot 12 \cdot 1000}} = 0.197$

$k_1\ \&\ k_2\ in\ parallel\ \rightarrow \omega_n = \sqrt{\frac{k_1}{m}} = \sqrt{\frac{45{,}000+1500}{32.2 \cdot 12 \cdot 1000}} = 0.394$

$k_1\ \&\ k_2\ in\ series \rightarrow \omega_n = \sqrt{\frac{k_1}{m}} = \sqrt{\frac{\frac{45{,}000 \cdot 15000}{45000+15000}}{32.2 \cdot 12 \cdot 1000}} = 0.1706$

We want the frequency ratio to be as large as possible therefore we want the natural frequency to be as small as possible

Therefore putting the two springs $k_1 \& k_2$ in series makes the most sense.