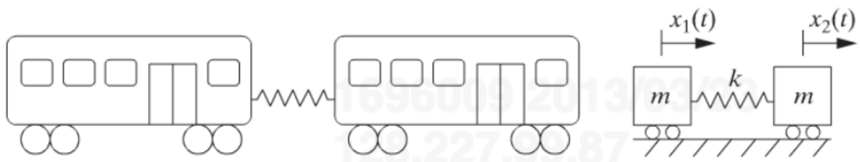
Homework 9

Vibrations

McGhee, Alexander

Problem 1

Two subway cars in the figure below have 2000 kg mass each and are connected by a coupler. The coupler can be modeled as a spring of stiffness $k = 280{,}000$ N/m. Suppose that the subway cars are given the initial position of $x_{10} = 0$, $x_{20} = 0.1$ m and initial velocities of $v_{10} = v_{20} = 0$. Calculate the natural frequencies and normalized mode shapes.



$$\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} [m] \quad , \quad \dot{\boldsymbol{x}}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Find the mass matrix and the inverse square root

$$\boldsymbol{M} = \begin{bmatrix} 2000 & 0 \\ 0 & 2000 \end{bmatrix}, \boldsymbol{M}^{1/2} = \begin{bmatrix} 44.72 & 0 \\ 0 & 44.72 \end{bmatrix}, \quad \boldsymbol{M}^{-1/2} = \begin{bmatrix} 0.0224 & 0 \\ 0 & 0.0224 \end{bmatrix}$$

Find the stiffness matrix

$$\boldsymbol{K} = \begin{bmatrix} -280{,}000 & 280{,}000 \\ 280{,}000 & -280{,}000 \end{bmatrix}$$

Find $\widetilde{K} = M^{-\frac{1}{2}} K M^{-\frac{1}{2}}$

$$\widetilde{K} = \begin{bmatrix} 0.0224 & 0 \\ 0 & 0.0224 \end{bmatrix} \begin{bmatrix} -280{,}000 & 280{,}000 \\ 280{,}000 & -280{,}000 \end{bmatrix} \begin{bmatrix} 0.0224 & 0 \\ 0 & 0.0224 \end{bmatrix} = \begin{bmatrix} -140 & 140 \\ 140 & -140 \end{bmatrix}$$

Use the relation $\det\left(\lambda I - \widetilde{K}\right) V = 0$

$$\det\left(\begin{bmatrix} \lambda + 140 & -140 \\ -140 & \lambda + 140 \end{bmatrix}\right) = 0$$

$$(\lambda + 140)(\lambda + 140) - 140^2 = 0$$

$$\lambda^2 + 280\lambda + 19600 - 19600 = 0$$

$$\lambda(\lambda + 280) = 0$$

$$\lambda_{1,2} = [0, -280]$$

$$\omega_1 = 0$$
$$\omega_2 = 16.7332\, i$$

First mode $\lambda = 0$

$$\begin{bmatrix} 140 & -140 \\ -140 & 140 \end{bmatrix} \begin{bmatrix} v_{11} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \quad V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Second mode $\lambda = -280$

$$\begin{bmatrix} -280 + 140 & -140 \\ -140 & -280 + 140 \end{bmatrix} \begin{bmatrix} v_{21} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \quad V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Normalize the modal vectors to obtain the modal matrix P

$$P = \begin{bmatrix} \frac{V_1}{\|V_1\|} & \frac{V_2}{\|V_2\|} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \end{bmatrix}$$

Calculate the mode shape $S = M^{-\frac{1}{2}}P$ & $S^{-1} = P^T M^{\frac{1}{2}}$

$$S = \begin{bmatrix} 0.0224 & 0 \\ 0 & 0.0224 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.0158 & -0.0158 \\ 0.0158 & 0.0158 \end{bmatrix}$$

$$S^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 44.72 & 0 \\ 0 & 44.72 \end{bmatrix} = \begin{bmatrix} 31.62 & 31.62 \\ -31.62 & 31.62 \end{bmatrix}$$

Calculate the modal initial conditions

$$\boldsymbol{r}_0 = S^{-1}\boldsymbol{x}_0 = \begin{bmatrix} 3.1623 \\ 3.1623 \end{bmatrix} \quad , \quad \dot{\boldsymbol{r}}_0 = S^{-1}\dot{\boldsymbol{x}}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Problem 2

Consider the matrix form of the equation of motion for a 2-DOF system as discussed in class:

$$M\ddot{\mathbf{x}} + K\mathbf{x} = 0$$

Let m_1=1 kg, m_2=4 kg, k_1 = 2 N/m, k_2 = 1 N/m.

(a) Determine the eigenvalues of the system.

Find the mass matrix and the inverse square root

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \boldsymbol{M}^{1/2} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \ \boldsymbol{M}^{-1/2} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

Find the stiffness matrix

$$\boldsymbol{K} = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

Find $\widetilde{K} = M^{-\frac{1}{2}} K M^{-\frac{1}{2}}$

$$\widetilde{K} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 3 & -1/2 \\ -1/2 & 1/4 \end{bmatrix}$$

Use the relation $\det(\lambda I - \widetilde{K})\, V = 0$

$$\det\left(\begin{bmatrix} \lambda - 3 & 1/2 \\ 1/2 & \lambda - 1/4 \end{bmatrix} \right) = 0$$

$$(\lambda - 3)(\lambda - 1/4) - 1/2^2 = 0$$

$$\lambda^2 - 3.25\lambda + 0.75 - 0.25 = 0$$

$$\lambda^2 - 3.25\lambda + 0.5 = 0$$

$$\lambda_{1,2} = [3.0881\,, 0.1619]$$

(b) Determine the related eigenvectors and normalize them.

First mode $\lambda = 3.0881$

$$\begin{bmatrix} 3.0881 - 3 & 1/2 \\ 1/2 & 3.0881 - 1/4 \end{bmatrix} \begin{bmatrix} v_{11} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow V_1 = \begin{bmatrix} -\frac{0.5}{0.0881} \\ 1 \end{bmatrix} \rightarrow V_1 = \begin{bmatrix} -5.6754 \\ 1 \end{bmatrix}$$

Second mode $\lambda = 0.1619$

$$\begin{bmatrix} 0.1619 - 3 & 1/2 \\ 1/2 & 0.1619 - 1/4 \end{bmatrix} \begin{bmatrix} v_{21} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow V_2 = \begin{bmatrix} -\frac{0.5}{-2.8381} \\ 1 \end{bmatrix} \rightarrow V_2 = \begin{bmatrix} 0.1762 \\ 1 \end{bmatrix}$$

Normalized eigenvectors

$$\hat{V}_1 = \begin{bmatrix} -0.9848 \\ 0.1762 \end{bmatrix} \quad , \quad V_2 = \begin{bmatrix} 0.1709 \\ 0.9699 \end{bmatrix}$$

Problem 3

Consider a similar system as in problem 2 with:

m_1= 1 kg, m_2= 4 kg, k_1= 240 N/m, k_2=300 N/m

Compute:

a) the natural frequencies b) the mode shapes c) the eigenvalues d) the eigenvectors
e) show that the mode shapes are not orthogonal f) show that the eigenvectors are orthogonal

a)

Find the mass matrix and the inverse square root

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \boldsymbol{M}^{1/2} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \boldsymbol{M}^{-1/2} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

Find the stiffness matrix

$$\boldsymbol{K} = \begin{bmatrix} 540 & -300 \\ -300 & 300 \end{bmatrix}$$

Find $\widetilde{K} = M^{-\frac{1}{2}} K M^{-\frac{1}{2}}$

$$\widetilde{K} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 540 & -150 \\ -150 & 75 \end{bmatrix}$$

Use the relation $\det\left(\lambda I - \widetilde{K}\right) V = 0$

$$\det\left(\begin{bmatrix}\lambda-540 & 150\\ 150 & \lambda-75\end{bmatrix}\right)=0$$

$$(\lambda-540)(\lambda-75)-150^2=0$$

$$\lambda^2-615\lambda+40500-22500=0$$

$$\lambda^2-615\lambda+18000=0$$

$$\lambda_{1,2}=[584.188\,,30.8120]$$

$$\omega_1=24.17$$
$$\omega_2=5.551$$

b)

First mode $\lambda=584.188$

$$\begin{bmatrix}584.188-540 & 150\\ 150 & 584.188-75\end{bmatrix}\begin{bmatrix}v_{11}\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\rightarrow V_1=\begin{bmatrix}-\dfrac{150}{44.188}\\ 1\end{bmatrix}\rightarrow V_1=\begin{bmatrix}-3.3946\\ 1\end{bmatrix}$$

Second mode $\lambda=30.8120$

$$\begin{bmatrix}30.8120-540 & 150\\ 150 & 30.8120-75\end{bmatrix}\begin{bmatrix}v_{21}\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\rightarrow V_2=\begin{bmatrix}-\dfrac{150}{-509.1880}\\ 1\end{bmatrix}\rightarrow V_2=\begin{bmatrix}0.2946\\ 1\end{bmatrix}$$

Normalized eigenvectors

$$\hat{V}_1=\begin{bmatrix}-0.9593\\ 0.2826\end{bmatrix}\quad,\quad\hat{V}_2=\begin{bmatrix}0.2826\\ 0.9592\end{bmatrix}$$

Mode shape

$$S=\begin{bmatrix}1 & 0\\ 0 & 1/2\end{bmatrix}\begin{bmatrix}-0.9593 & 0.2826\\ 0.2826 & 0.9592\end{bmatrix}=\begin{bmatrix}-0.9593 & 0.2826\\ 0.1413 & 0.4796\end{bmatrix}$$

Mode shapes are not orthogonal

$$S_1\cdot S_2=(-0.9593\cdot 0.2826)+(0.1413\cdot 0.4796)=0.3389$$

Because the dot product is not 0 there is a component of one mode shape in the direction of the other.

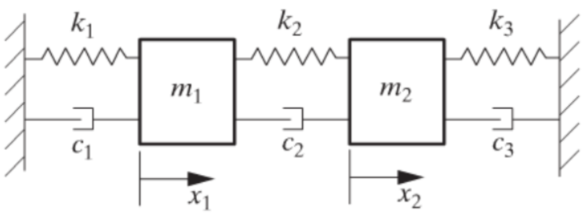
Eigenvectors are orthogonal

$$V_1\cdot V_2=(-3.3946\cdot 0.2946)+(1\cdot 1)=0$$

Since the dot product is 0, the two vectors must be orthogonal.

Problem 4

a) Calculate the vibration response for the system below, let $c_1 = c_2 = c_3 = 0$ and initial conditions are $\mathbf{x}(0) = \mathbf{O}, \dot{\mathbf{x}}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$.



$m_1 = 9$ kg, $m_2 = 1$ kg, $k_1 = 24$ N/m, $k_2 = 3$ N/m, $k_3 = 3$ N/m

Find the mass matrix and the inverse square root

$$\boldsymbol{M} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}, \boldsymbol{M}^{1/2} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{M}^{-1/2} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}$$

Find the stiffness matrix

$$\boldsymbol{K} = \begin{bmatrix} 27 & -3 \\ -3 & 9 \end{bmatrix}$$

Find $\widetilde{K} = M^{-\frac{1}{2}} K M^{-\frac{1}{2}}$

$$\widetilde{K} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 27 & -3 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 9 \end{bmatrix}$$

Use the relation $\det(\lambda I - \widetilde{K}) V = 0$

$$\det\left(\begin{bmatrix} \lambda - 3 & 1 \\ 1 & \lambda - 9 \end{bmatrix}\right) = 0$$

$$(\lambda - 3)(\lambda - 9) - 1^2 = 0$$

$$\lambda^2 - 12\lambda + 27 - 1 = 0$$

$$\lambda^2 - 12\lambda + 26 = 0$$

$$\lambda_{1,2} = [9.1623, 2.8377]$$

First mode $\lambda = 9.1623$

$$\begin{bmatrix} 9.1623 - 3 & 1 \\ 1 & 9.1623 - 9 \end{bmatrix} \begin{bmatrix} v_{11} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow V_1 = \begin{bmatrix} -\dfrac{1}{6.1623} \\ 1 \end{bmatrix} \rightarrow V_1 = \begin{bmatrix} -0.1623 \\ 1 \end{bmatrix}$$

Second mode $\lambda = 2.8377$

$$\begin{bmatrix} 2.8377 - 3 & 1 \\ 1 & 2.8377 - 9 \end{bmatrix} \begin{bmatrix} v_{21} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow V_2 = \begin{bmatrix} \dfrac{1}{0.1623} \\ 1 \end{bmatrix} \rightarrow V_2 = \begin{bmatrix} 6.1614 \\ 1 \end{bmatrix}$$

Normalize the modal vectors to obtain the modal matrix P

$$P = \left[\frac{V_1}{\|V_1\|} \quad \frac{V_2}{\|V_2\|}\right] = \left[\begin{bmatrix} -0.1602 & 0.9871 \\ 0.9871 & 0.1602 \end{bmatrix}\right]$$

Calculate the mode shape $S = M^{-\frac{1}{2}}P \quad \& \quad S^{-1} = P^T M^{\frac{1}{2}}$

$$S = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -0.1602 & 0.9871 \\ 0.9871 & 0.1602 \end{bmatrix} = \begin{bmatrix} -0.0534 & 0.3292 \\ 0.9871 & 0.1602 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} -0.1602 & 0.9871 \\ 0.9871 & 0.1602 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.4806 & 0.9871 \\ 2.9612 & 0.1602 \end{bmatrix}$$

Calculate the modal initial conditions

$$\boldsymbol{r}_0 = S^{-1}\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad , \quad \dot{\boldsymbol{r}}_0 = S^{-1}\dot{\boldsymbol{x}}_0 = \begin{bmatrix} -0.4806 \\ 2.9612 \end{bmatrix}$$

Using the differential equation

$$\ddot{r} + \Lambda r = 0$$

We get

$$r_1(t) = \sqrt{\frac{(0.4806)^2}{9.1623}} \sin\left(\sqrt{9.1623}\, t\right) \quad \rightarrow \quad 0.1588 \sin(3.0269\, t)$$

$$r_2(t) = \sqrt{\frac{(2.9612)^2}{2.8377}} \sin\left(\sqrt{2.8377}\, t\right) \quad \rightarrow \quad 1.7576 \sin(1.6845\, t)$$

b) Plot and print the response for both masses using MATLAB.