EML 5526 Finite Element Analysis & application Alexander McGhee

Homework 1

Problem 1) Rewrite the following equations using index notation

a) The equilibrium equations for static 2D elastic structures:

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$
$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} = 0$$

Solution

$$\text{define}: \quad x_i \; \{\text{x, y} = 1{,}2\}$$
$$\frac{\partial \tau_{ij}}{\partial x_j} = 0 \quad \{i = 1{,}2 \;\; \& \;\; j = 1{,}2\}$$
$$Or\ more\ compact$$
$$\tau_{ij,j} = 0 \quad \{i = 1{,}2 \;\; \& \;\; j = 1{,}2\}$$

Proof

When $i = 1 \;\; \& \;\; j = 1$

$$\frac{\partial \tau_{xx}}{\partial x}$$

When $i = 1 \;\; \& \;\; j = 2$

$$\frac{\partial \tau_{xy}}{\partial y}$$

Due to the implied sum

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

When $i = 2 \;\; \& \;\; j = 1$

$$\frac{\partial \tau_{yx}}{\partial x}$$

When $i = 2 \;\; \& \;\; j = 2$

$$\frac{\partial \tau_{yy}}{\partial y}$$

Due to the implied sum

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} = 0$$

b) The convection diffusion equation:

$$\mathbf{U} \cdot \nabla T = \alpha \nabla^2 T$$

Where, T is a scalar field, **U** is a 3D vector field and α is a scalar constant

Solution

$$U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix} = U_{ij} \quad \{i, j = 1,2,3\}$$

$$\nabla T = \frac{\partial T}{\partial x_1}\hat{x}_1 + \frac{\partial T}{\partial x_2}\hat{x}_2 + \frac{\partial T}{\partial x_3}\hat{x}_3 = \begin{Bmatrix} \frac{\partial T}{\partial x_1} \\ \frac{\partial T}{\partial x_2} \\ \frac{\partial T}{\partial x_3} \end{Bmatrix} = \frac{\partial T}{\partial x_i}\hat{x}_i = T_{,i}\hat{x}_i = T_{,i} \ \{i = 1,2,3\}$$

$$\alpha \nabla^2 T = \alpha\left(\frac{\partial^2 T}{\partial x_1^2}\hat{x}_1 + \frac{\partial^2 T}{\partial x_2^2}\hat{x}_2 + \frac{\partial^2 T}{\partial x_3^2}\hat{x}_3\right) = \alpha \begin{Bmatrix} \frac{\partial^2 T}{\partial x_1^2} \\ \frac{\partial^2 T}{\partial x_2^2} \\ \frac{\partial^2 T}{\partial x_3^2} \end{Bmatrix} = \alpha \frac{\partial^2 T}{\partial x_i^2}\hat{x}_i = \alpha T_{,ii}\hat{x}_i = \alpha T_{,ii} \ \{i = 1,2,3\}$$

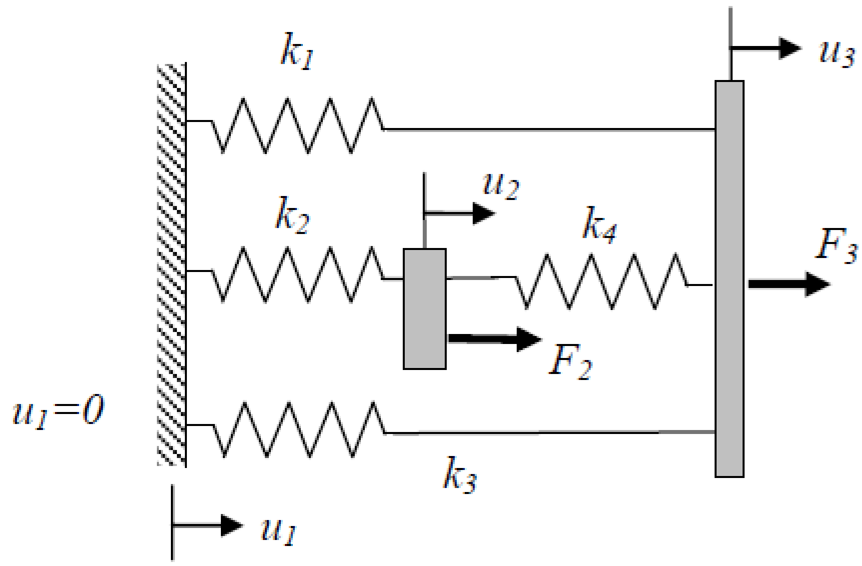
$$U_{ij} \cdot \frac{\partial T}{\partial x_i} = \alpha \frac{\partial^2 T}{\partial {x_i}^2} \quad \{i = 1,2,3\}$$

$$U_{ij} \cdot T_{,i} = \alpha T_{,ii} \quad \{i = 1,2,3\}$$

Homework 1

Problem 2)

The rigid blocks shown in Fig. 1 are connected by linear springs. Assume that only horizontal displacements are allowed. (a) Write the element equations [**K**$_j$]{**D**$_j$}={**R**$_j$} for each spring element assuming the spring stiffness is k_j (b) Assemble the element equations to form a global system equation [**K**]{**X**}={**F**}



Connection Table

Element	LN 1	LN 2
1	1	3
2	1	2
3	1	3
4	2	3

a)

$$k_1\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_3\end{Bmatrix}=\begin{Bmatrix}-F_{k1}\\ F_{k1}\end{Bmatrix}$$
$$k_2\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}=\begin{Bmatrix}-F_{k2}\\ F_{k2}\end{Bmatrix}$$
$$k_3\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_3\end{Bmatrix}=\begin{Bmatrix}-F_{k3}\\ F_{k3}\end{Bmatrix}$$
$$k_4\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}-F_{k4}\\ F_{k4}\end{Bmatrix}$$

b)

Use Connection Table to create matrix

$$\begin{bmatrix}k_1+k_2+k_3 & -k_2 & -k_1-k_3\\ -k_2 & k_2+k_4 & -k_4\\ -k_1-k_3 & -k_4 & k_1+k_3+k_4\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}-F_{k1}-F_{k2}-F_{k3}\\ F_{k2}-F_{k4}\\ F_{k1}+F_{k3}+F_{k4}\end{Bmatrix}$$

Use Boundary conditions

$$u_1=0$$

$$\begin{bmatrix} k_2 + k_4 & -k_4 \\ -k_4 & k_1 + k_3 + k_4 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_{k2} - F_{k4} \\ +F_{k1} + F_{k3} + F_{k4} \end{Bmatrix}$$

Force Balance

$$F_3 = F_{k1} + F_{k3} + F_{k4}$$
$$F_2 = F_{k2} - F_{k4}$$

Substitution

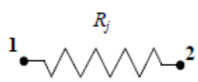
$$\begin{bmatrix} k_2 + k_4 & -k_4 \\ -k_4 & k_1 + k_3 + k_4 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

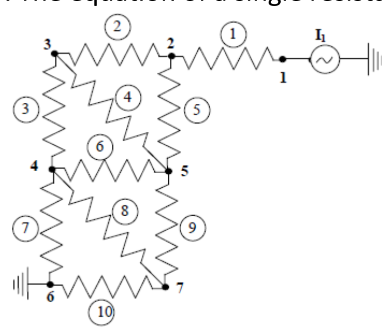
Solution

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} \dfrac{F_2(k_1 + k_3 + k_4)}{k_2(k_1 + k_3) + k_4(k_1 + k_2 + k_3)} + \dfrac{F_3 k_4}{k_1(k_2 + k_4) + k_2 k_3 + k_4(k_2 + k_3)} \\ \dfrac{F_3(k_2 + k_4)}{k_2(k_1 + k_3) + k_4(k_1 + k_2 + k_3)} + \dfrac{F_2 k_4}{k_1(k_2 + k_4) + k_2 k_3 + k_4(k_2 + k_3)} \end{Bmatrix}$$

Problem 3)

Consider the electrical network shown in Fig.3. The equation of a single resistance can be written as:





a) Write the stiffness matrix for a single resistor element, and express the equation as:

$$[K]\{V\} = \{I\}$$

For a resistor network The "stiffness" relates to $\frac{1}{R_j}$

The "displacement" relates to $(V_1 - V_2)$

The "Force" relates to current I

Therefor the stiffness matrix would be

$$[K] = \frac{1}{Rj}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} V_{LN1} \\ V_{LN2} \end{Bmatrix} = \begin{Bmatrix} I_j \\ -I_j \end{Bmatrix}$$

b) Connectivity Table

Element	LN 1	LN 2
1	1	2
2	2	3
3	3	4
4	3	5
5	2	5
6	4	5
7	4	6
8	4	7
9	5	7
10	6	7

c) Assemble the stiffness matrix and write the system of equations

$$\begin{bmatrix} R_1^{-1} & -R_1^{-1}+R_5^{-1} & 0 & 0 & -R_5^{-1} & 0 & 0 \\ -R_1^{-1} & R_1^{-1}+R_2^{-1} & -R_2^{-1} & 0 & 0 & 0 & 0 \\ 0 & -R_2^{-1} & R_2^{-1}+R_3^{-1}+R_4^{-1} & -R_3^{-1} & -R_4^{-1} & 0 & 0 \\ 0 & 0 & -R_3^{-1} & R_3^{-1}+R_6^{-1}+R_7^{-1}+R_8^{-1} & -R_6^{-1} & -R_7^{-1} & -R_8^{-1} \\ 0 & -R_5^{-1} & -R_4^{-1} & -R_6^{-1} & R_4^{-1}+R_5^{-1}+R_6^{-1}+R_9^{-1} & 0 & -R_9^{-1} \\ 0 & 0 & 0 & -R_7^{-1} & 0 & R_7^{-1}+R_{10}^{-1} & -R_{10}^{-1} \\ 0 & 0 & 0 & -R_8^{-1} & -R_9^{-1} & -R_{10}^{-1} & R_8^{-1}+R_9^{-1}+R_{10}^{-1} \end{bmatrix} \cdot \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{Bmatrix} = \begin{Bmatrix} I_1 \\ -I_1+I_2+I_5 \\ -I_2+I_3+I_4 \\ -I_3+I_6+I_7+I_8 \\ -I_4-I_5-I_6+I_9 \\ -I_7+I_{10} \\ -I_8-I_9-I_{10} \end{Bmatrix}$$

d) Apply boundary conditions

Voltage at node 6 = 0

$$V_6 = 0$$

The sum of all currents at a node is 0

$$\begin{bmatrix} R_1^{-1} & -R_1^{-1} & 0 & 0 & 0 & 0 \\ -R_1^{-1} & R_1^{-1}+R_2^{-1}+R_5^{-1} & -R_2^{-1} & 0 & -R_5^{-1} & 0 \\ 0 & -R_2^{-1} & R_2^{-1}+R_3^{-1}+R_4^{-1} & -R_3^{-1} & -R_4^{-1} & 0 \\ 0 & 0 & -R_3^{-1} & R_3^{-1}+R_6^{-1}+R_7^{-1}+R_8^{-1} & -R_6^{-1} & -R_8^{-1} \\ 0 & -R_5^{-1} & -R_4^{-1} & -R_6^{-1} & R_4^{-1}+R_5^{-1}+R_6^{-1}+R_9^{-1} & -R_9^{-1} \\ 0 & 0 & 0 & -R_8^{-1} & -R_9^{-1} & R_8^{-1}+R_9^{-1}+R_{10}^{-1} \end{bmatrix} \cdot \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_7 \end{Bmatrix} = \begin{Bmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

e) Using MATLAB with $R_i = 200$, $I = 0.1\ (A)$ to solve I get

$$\begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_7 \end{Bmatrix} = \begin{Bmatrix} 47.27 \\ 27.27 \\ 18.18 \\ 10.9 \\ 16.36 \\ 9.09 \end{Bmatrix}$$

f) Compute the current through elements 7 & 10. What is the current flowing to the ground at node 6.

$$I = \frac{V}{R}$$

$$I_{10} = \frac{V_7}{R} = \frac{9.09}{200} = 0.04545\ (A)$$

$$I_7 = \frac{V_4}{R} = \frac{10.9}{200} = 0.0545\ (A)$$

Because the current going to the ground must be a combination of the other two currents coming into node 6, the current flowing to ground is

$$I_{Ground} = 0.04545 + 0.0545 \cong 0.1$$