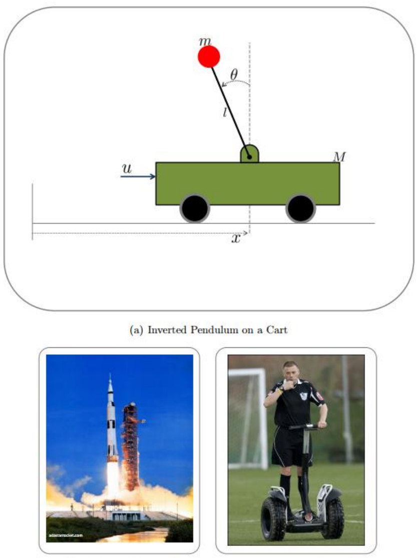
EML 4312: Control of mechanical engineering systems Alexander McGhee

Homework 1

Consider the following dynamical system

$$\ddot{x} + \frac{1}{M_t - mkc^2\theta}\left(ml\dot{\theta}^2 \sin(\theta) - mkgsin(\theta)\cos(\theta) + \eta\dot{x} + \gamma ml\dot{\theta}\cos(\theta) - u\right) = 0$$

$$\ddot{\theta} + \frac{m}{J_t(M_t - mkc^2\theta)}\left(ml^2\dot{\theta}^2 \sin(\theta)\cos(\theta) - M_t glsin(\theta) + l\eta\dot{x}\cos(\theta) + \gamma\frac{M_t}{m}\dot{\theta}\cos(\theta) - ulcos(\theta)\right) = 0$$



(a) Inverted Pendulum on a Cart

(b) A Rocket Lifting Off (c) Segway Personal Transport System

The above equations model an inverted pendulum being balanced by a moving cart. The idea is to keep the pendulum upright by applying a control input (u) on the cart. Two examples of such "balance systems" are shown in Figs.1(b) and 1(c). Note that in both these physical examples, a control force is applied on the bottom portion of the system to keep it upright.

For Equation 1 and 2

- m= mass of pendulum
- M= mass of cart
- $m + M = M_t =$ total mass
- J is the moment of inertia $J_t = ml^2 + J$
- η & γ are damping coefficients
- $k = \frac{ml^2}{J_t}$.

Problem 1) __(10 points)__

a) Identify the states of the above dynamical system.
   - x
   - θ
   - $\dot{x}$
   - $\dot{\theta}$

b) Express the system in state-space for

Define

$x_1 = \theta$

$x_2 = \dot{\theta}$

$x_3 = x$

$x_4 = \dot{x}$

Derive and equate

$\dot{x}_1 = x_2 = \dot{\theta}$

$$\dot{x}_2 = \ddot{\theta} = -\frac{m}{J_t(M_t - mk\,cos^2(x_1))}\left(ml^2x_2\sin(x_1)\cos(x_1) - M_t glsin(x_1) + l\eta x_4\cos(x_1) + \gamma\frac{M_t}{m}x_2\cos(x_1) - ulcos(x_1)\right)$$

$\dot{x}_3 = x_4 = \dot{x}$

$$\dot{x}_4 = \ddot{x} = -\frac{1}{M_t - mk\,cos^2(x_1)}\left(mlx_2^2\sin(x_1) - mkgsin(x_1)\cos(x_1) + \eta x_4 + \gamma mlx_2\cos(x_1) - u\right)$$

c) Is the system linear?
If $\boldsymbol{f}(t, x_1, x_2, x_3, x_4, u) = \boldsymbol{A}x + \boldsymbol{B}u$ then the system is linear

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{?_1}{?_2} & \left(ml^2 + \frac{\gamma M_t}{m}\right)\frac{?_1}{?_2} & 0 & l\eta \cdot \frac{?_1}{?_2} \\ 0 & 0 & 0 & 1 \\ \frac{?_3}{?_4} & \frac{?_3}{?_4} & 0 & \eta\frac{?_3}{?_4} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} + \begin{Bmatrix} 0 \\ ? \\ 0 \\ \frac{1}{?} \end{Bmatrix} u$$

Because >1 of the elements of the above matrix cannot be solved as a constant coefficient of x_1, x_2, x_3 or x_4 this system is nonlinear.

d) Set $u = 0$ (no control). Is $\boldsymbol{x}^*(t) = 0$ a steady-state reference trajectory of this system?

Step 1: call $\boldsymbol{x}^* = \{x_1^*, x_2^*, x_3^*, x_4^*\}^T = \{0,0,0,0\}^T$ the reference point for the Taylor series approximation

Step 2: define a perturbation $\boldsymbol{\delta} = (\boldsymbol{x} - \boldsymbol{x}^*) \rightarrow \ \boldsymbol{x} = \boldsymbol{\delta} + \boldsymbol{x}^* = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix} + \begin{Bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{Bmatrix}$

Step 3: do math

$$\widehat{\boldsymbol{y}}_L(\boldsymbol{x}) = f(\boldsymbol{x}^*) + \left[\frac{\delta}{\delta \boldsymbol{x}}\left(f(\boldsymbol{x}^*)\right)\right]^T \boldsymbol{\delta}$$

$$\frac{\delta \dot{x}_2}{\delta x_1} = \frac{m\left(Mgl\cos(x_1) - l^2 m x_2 \cos(x_1)^2 - lu\sin(x_1) + l^2 m x_2 \sin(x_1)^2 + l n x_4 \sin(x_1) + \frac{M x_2 \gamma \sin(x_1)}{m}\right)}{J(M - km\cos(x_1)^2)} + \frac{2km^2\cos(x_1)\sin(x_1)\left(lnx_4\cos(x_1) - Mgl\sin(x_1) - lu\cos(x_1) + \frac{Mx_2\gamma\cos(x_1)}{m} + l^2 m x_2 \cos(x_1)\sin(x_1)\right)}{J(M - km\cos(x_1)^2)^2} = \frac{mMgl}{J(M - km)}$$

$$\frac{\delta \dot{x}_2}{\delta x_2} = -\frac{m\left(l^2 m cos(x_1)\sin(x_1) + \frac{M\gamma cos(x_1)}{m}\right)}{J(M - kmcos(x_1)^2)} = -\frac{M\gamma}{J(M - km)}$$

$$\frac{\delta \dot{x}_2}{\delta x_3} = 0$$

$$\frac{\delta \dot{x}_2}{\delta x_4} = -\frac{lmncos(x_1)}{J(M - kmcos(x_1)^2)} = -\frac{lmn}{J(M - km)}$$

$$\frac{\delta \dot{x}_4}{\delta x_1} = \frac{2kmcos(x_1)\sin(x_1)\left(msin(x_1)l^{2x_2^2} + mycos(x_1)lx_2 - u + nx_4 - gkmcos(x_1)\sin(x_1)\right)}{(M - kmcos(x_1)^2)^2} - \frac{m\,l^2 x_2^2\cos(x_1) - myl\,x_2 sin(x_1) - gkmcos(x_1)^2 + gkmsin(x_1)^2}{M - kmcos(x_1)^2} = \frac{gkm}{M - km}$$

$$\frac{\delta \dot{x}_4}{\delta x_2} = -\frac{2m\,x_2 sin(x_1)\,l^2 + mycos(x_1)l}{M - kmcos(x_1)^2} = -\frac{m\gamma l}{M - km}$$

$$\frac{\delta \dot{x}_4}{\delta x_3} = 0$$

$$\frac{\delta \dot{x}_4}{\delta x_4} = -\frac{n}{M - kmcos(x_1)^2} = -\frac{n}{M - km}$$

Homework 1

$$\hat{y}_L \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{mMgl}{J(M-km)} & -\frac{M\gamma}{J(M-km)} & 0 & -\frac{lmn}{J(M-km)} \\ 0 & 0 & 0 & 1 \\ \frac{gkm}{M-km} & -\frac{m\gamma l}{M-km} & 0 & -\frac{n}{M-km} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix}$$

Because $\hat{y}_L$ is a constant function of time, and $f(\boldsymbol{x}^*) = \boldsymbol{0}$ then, yes $\boldsymbol{x}^*(t) = \boldsymbol{0}$ is a steady-state reference trajectory.

Problem 4 **(10 points)**

Consider the following nonlinear function:

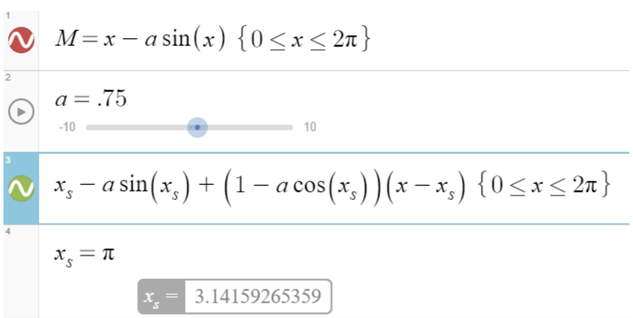
$$M = E - e\, sin(E);\ \mathrm{E} \in [0, 2\pi$$

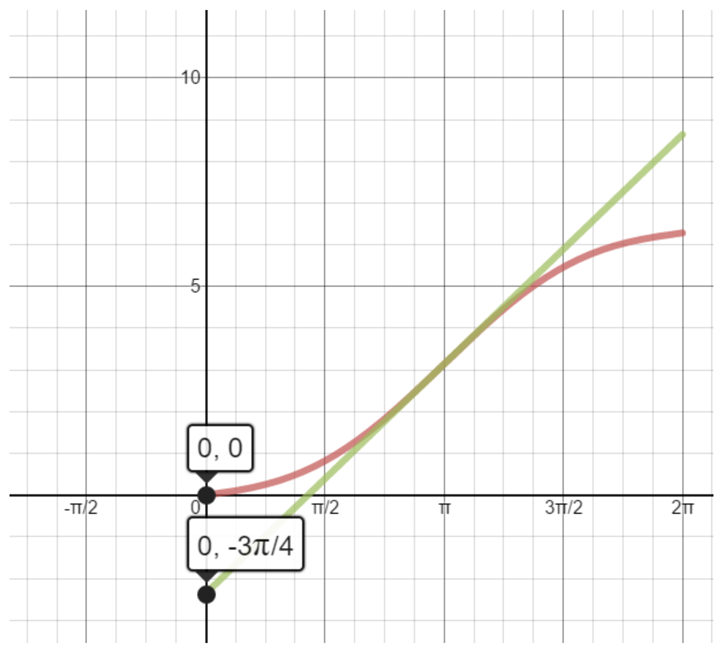
This is called Kepler's equation for elliptic orbits in space: a nonlinear relationship between the so-called mean anomaly (M) and eccentric anomaly (E). It has remained unsolved since it was first derived, which was over 400 years ago. The quantity e is a constant and represents the eccentricity of the elliptic orbit. Let e = 0.75 for this problem

a) Determine a linear approximation, $\widehat{M}_L(E)$, Use your judgment to pick a reference point. Graph the nonlinear function and its linear approximation in a single figure for comparison.

Linear approximation
Step 1 : Pick a reference point
$x^*(t) = pi$: this is a good point because the linear approximation will have symmetric error.
Step 2 :
find $f(x^*) = \pi - 0.75\sin(\pi) = \pi$
find $f'(x^*) = 1 - 0.75\cos(\pi) = 1 + 0.75$
Linear Taylor series is the first two terms neglecting HOT

$$\widehat{M}_L = \pi + (1.75)\delta = \pi + (1.75)(x - \pi)$$





b) What is the maximum error resulting from linearization in the domain of interest?

The max error is $\frac{3\pi}{4} = 2.3561$
Occurring at $x = a(2\pi)\ \ \{a \in integer\}$

c) Add in the quadratic term to obtain the quadratic approximation, $\widehat{M}_Q(E)$ and comment on the change in maximum error.
Find the 2nd order term

$$f''(x^*) = 0.75sin(\pi) = 0$$

Because the 2nd derivative is equal to 0 the change in maximum error between the 1st and 2nd order term is 0

Odd order terms do play a role however, because *sine* is an odd function

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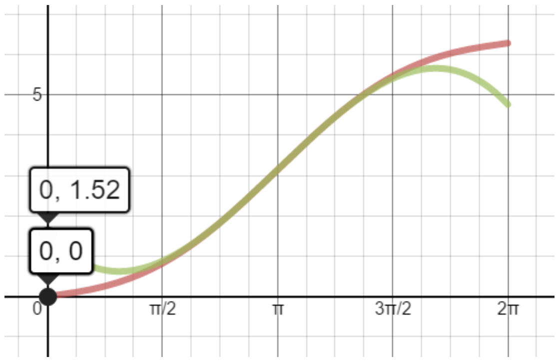
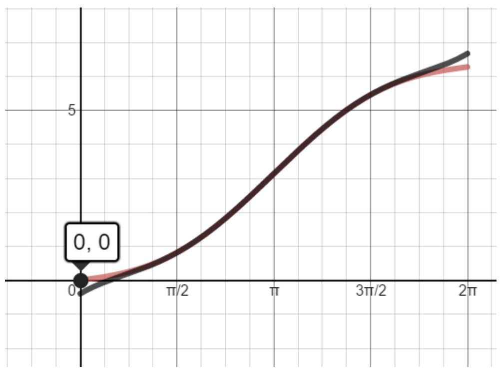


3rd order term included



5th order term included